Gravitation and Satellites

Newton's Law of Universal Gravitation

Any two masses will experience a force of attraction. The magnitude of this force is given as:

$$F = G\frac{m_1 m_2}{r^2}$$

m_1 and m_2 are the masses of the objects
r is the distance between their centres
G is the constant of universal gravitation, $G = 6.67\times10^{-11}\text{Nm}^2\text{kg}^{-2}$

The magnitude of the force is proportional to the masses

- increasing either mass will increase the force experienced
- increasing both will significantly increase the force.

The magnitude of the force is inversely proportional to the radius squared

- decreasing r (bringing the objects closer together) will significantly increase the force.

Note that the gravitational forces are consistent with Newton's third law.
m_1 feels a force in the direction of m_2, and m_2 feels an equal magnitude force in the opposite direction (towards m_1).

If the bodies have different masses, the magnitude of acceleration is not equal on both, since $a = \frac{F}{m}$
Objects with smaller mass experience greater acceleration, and objects with greater mass experience less acceleration.

Acceleration due to gravity

Any mass m near another mass M will accelerate towards M. This acceleration due to gravity (denoted g) is the following for some m and r:

$$F = G\frac{mM}{r^2} \quad , \quad g = \frac{F}{m}$$

$$\therefore g = \frac{G\frac{mM}{r^2}}{m}$$

$$= G\frac{mM}{r^2}\times\frac{1}{m}$$

$$g = G\frac{M}{r^2}$$

The acceleration of any object due to gravity depends only on the *other* object's mass and the distance between.

For example the acceleration of a cannonball m towards the Earth M depends only on the Earth's mass M and the distance from the cannonball to the centre of the earth, r.

Satellites in Circular Orbits

A satellite orbiting a planet has its centripetal acceleration provided by the force of gravitation. Since we know the equation for centripetal acceleration and the equation for acceleration due to gravity, we can rearrange to find the speed of the satellite.

$$a_c = g$$
$$\therefore \frac{v^2}{r} = \frac{GM}{r^2}$$
$$\therefore v^2 = \frac{GM}{r} \quad \{\text{multiplying both sides by } r\}$$
$$\therefore v = \sqrt{\frac{GM}{r}}$$

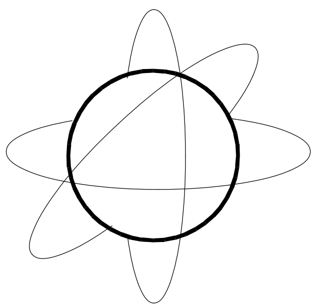
The period *T* of the satellite is the time it takes to complete one orbit (circumference), so

$$s = vt \quad \therefore t = \frac{s}{v}$$
$$\therefore T = \frac{2\pi r}{v}$$
$$\therefore T = \frac{2\pi r}{\sqrt{\frac{GM}{r}}}$$
$$\therefore T^2 = \frac{4\pi^2 r^2}{\frac{GM}{r}} \quad \{\text{squaring both sides to simplify}\}$$
$$\therefore T^2 = \frac{4\pi^2 r^2}{1} \times \frac{r}{GM} \quad \left\{\text{since } \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c}\right\}$$
$$\therefore T^2 = \frac{4\pi^2 r^3}{GM}$$

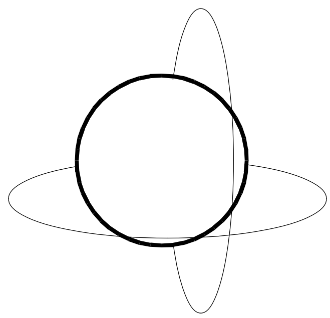
Notice that the speed (and therefore the period which depends on the speed) of a satellite in a circular orbit depends *only* on the radius of the orbit and mass of the body it is orbiting (*not* the mass of the satellite itself).

Possible orbits of satellites

The centre of the orbit circle of any satellite *must* be at the same position as the centre of the planet it orbits, since the centripetal acceleration will *always* be towards the centre of the planet due to being provided by gravitational attraction which is between the centres of mass.



Possible orbits



Impossible orbits

Geostationary satellites

Also known as geosynchronous satellites, these are 'stationary' with respect to the Earth's surface. That is, a geostationary satellite orbits the Earth in the same direction and with the same period as Earth, so that it is always above the same position on the Earth's surface at any given time.

Since the satellite has the same period as Earth (24 hours or 8.64×10^4s), we can calculate its orbital speed and radius:

$$v = \sqrt{\frac{GM}{r}}$$

$$\therefore v^2 = \frac{GM}{r}$$

We know that $T = \frac{2\pi r}{v} \quad \therefore v = \frac{2\pi r}{T}$

$$\therefore \left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r}$$

$$\therefore \frac{4\pi^2 r^2}{T^2} = \frac{GM}{r}$$

$$\therefore 4\pi^2 r^3 = T^2 GM \quad \{\text{multiplying both sides by } T^2 r\}$$

$$\therefore r^3 = \frac{T^2 GM}{4\pi^2}$$

$$\therefore r = \sqrt[3]{\frac{T^2 GM}{4\pi^2}}$$

$$= \sqrt[3]{\frac{(8.64\times10^4)^2\times6.67\times10^{-11}\times5.98\times10^{24}}{4\pi^2}}$$

$$= 4.22\times10^7\,\text{m (3 s.f.)}$$

So a geostationary satellite can only be at one height; 4.22×10^7 m from the centre of the Earth (3.59×10^7 m from the Earth's surface since the Earth's radius is 6.38×10^6 m).

This also means every geostationary satellite must be going a particular speed:

$$v = \sqrt{\frac{GM}{r}}$$

$$= \sqrt{\frac{6.67\times10^{-11}\times5.98\times10^{24}}{4.23\times10^7}}$$

$$= 3.07\times10^3\,\text{ms}^{-1}\ \text{(3 s.f.)}$$

A geostationary satellite must *always* be over the equator since it must travel east to west with the Earth and also have the centre of its circle of motion coincide with the centre of the Earth.

Launching low-altitude equatorial-orbit satellites

The earth rotates from west to east with a speed of about 465ms^{-1} at the surface. If we launch a satellite into a west-to-east equatorial orbit, we save on fuel since the satellite will already have 465ms^{-1} of the speed it needs to maintain its orbit. If we want the satellite to go east-to-west then we are at a disadvantage since the Earth will give it an initial speed in the wrong direction.

This is especially important for lower orbits since the speed required to orbit gets higher as the orbit gets lower, according to $v = \sqrt{\frac{GM}{r}}$

Satellites in low-altitude polar orbits

A low-altitude polar orbit is useful for meteorology and surveillance. Being closer to the Earth means that information is clearer, and being in a polar orbit (north-to-south-to-north-etc) means the Earth will rotate (west to east) underneath the satellite, so it covers the whole Earth bit by bit. A low altitude polar orbit makes about 14 orbits each day, surveying the Earth in a series of north-south and south-north strips.

In fact, if the satellite is angled just right (about 8 degrees off exactly north-south rotation) it will return to the same position at the same time each day.

Note: An orbit of anything up to as high as 3000km is considered low in satellite terms. This is because satellites need to be launched to heights far above the Earth's surface to avoid air friction and also to reduce the speed required to stay in orbit (since speed is inversely proportional to the radius of circular motion). A medium orbit is 3000 to 12000 km, and anything above that is considered high.