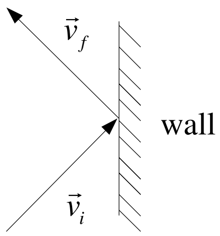
Momentum in Two Dimensions

Vector form of Newton's second law

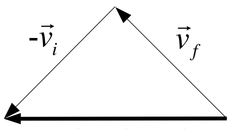
The force and acceleration in Newton's second law are vectors rather than scalars. That is, they have a direction as well as a magnitude.

$\vec{F} = m\vec{a}$

Consider an object bouncing off a fixed surface, where the final speed is equal to the initial speed.



$\Delta\vec{v}$ can be found by vector subtraction, as shown below.



$\Delta\vec{v} = \vec{v}_f - \vec{v}_i$

$\vec{a}_{\text{ave}} = \dfrac{\Delta\vec{v}}{\Delta t}$ will give the average acceleration of the ball. This will be in the same direction as $\Delta\vec{v}$, which in this case is directly to the left.

$\vec{F}_{\text{ave}} = m\vec{a}_{\text{ave}}$ will give the average force exerted on the object by the wall.

In this case, the force on the object by the wall is to the left. According to Newton's third law, the force exerted on the wall by the object is equal in magnitude and to the right.

Newton's second law in terms of momentum

The momentum $\vec{p}$ of an object is its mass multiplied by its velocity, so $\vec{p} = m\vec{v}$.
An object's momentum is in the same direction as its velocity.
Momentum is usually given in units of kg m s^{-1}.

$$\vec{F} = m\vec{a} = m\frac{\vec{v}_f - \vec{v}_i}{\Delta t} = \frac{m\vec{v}_f - m\vec{v}_i}{\Delta t}$$

$$\therefore \vec{F} = \frac{\Delta m\vec{v}}{\Delta t}$$

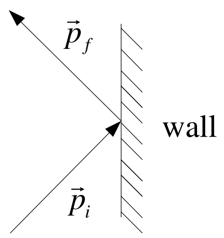
$$\therefore \vec{F} = \frac{\Delta\vec{p}}{\Delta t}$$

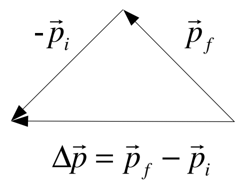
The above equation only applies if:

- the mass is constant, and
- force and acceleration are considered to be constant (or an average is being used).

Vector diagrams for momentum

Vectors representing momentum can be added and subtracted just like any other vector, for example a ball bouncing off a wall with no change in speed will therefore have no change in magnitude of momentum:



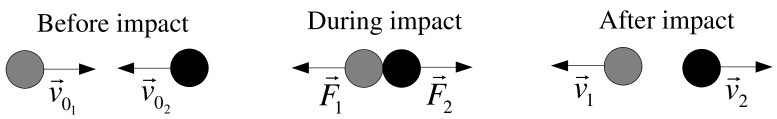


The force can be found using the relationship $\vec{F} = \dfrac{\Delta\vec{p}}{\Delta t}$. The force will be in the same direction as $\Delta\vec{p}$.

Law of conservation of momentum

During a collision between two or more bodies, the bodies' velocities change (and therefore their momentums change since $\vec{p} = m\vec{v}$).
However, the total momentum of the system always remains constant.

This can be seen for two objects by combining Newton's second law in terms of momentum with Newton's third law. During impact each body experiences an equal force in opposite directions:



$$\vec{F}_1 = -\vec{F}_2 \quad \{\text{Newton's third law}\}$$
$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t} \quad \{\text{Newton's second law in terms of momentum}\}$$
$$\therefore \frac{\Delta\vec{p}_1}{\Delta t} = -\frac{\Delta\vec{p}_2}{\Delta t}$$
$$\therefore \Delta\vec{p}_1 = -\Delta\vec{p}_2$$
$$\therefore \Delta\vec{p}_1 + \Delta\vec{p}_2 = 0$$

That is, the total change in momentum is zero.

Formally, the Law of Conservation of Momentum can be given as:
The total momentum of any number of objects remains unchanged in the absence of external forces.
(“Any number of objects” is also known as a “system”).

Multi-image puck collision analysis

A collision between two pucks can be considered by recording a multi-image photograph.

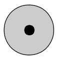
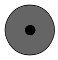
If the flash rate (time between exposures) is constant, and one of the masses is an integral multiple of the other, the photograph can be used to compare total momentum before and after the collision.

Ignore the scale of the photograph, the flash rate, and the actual masses.

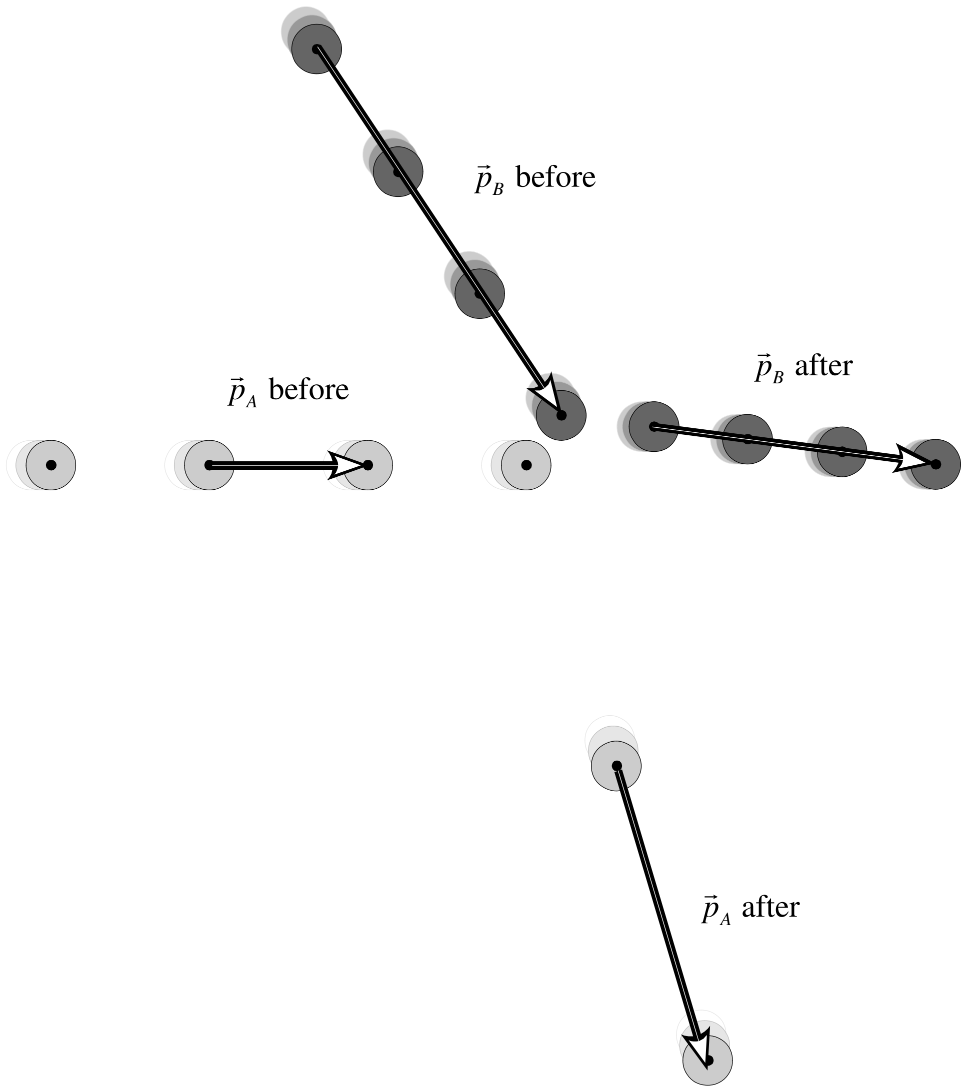
1. Use the distance moved between exposures to make velocity vectors before and after.
   Try to avoid using exposures at the collision point as they are not exclusively before or after.
2. Multiply these by their relative masses to get the momentum vectors.
   Since the objects are at constant speed before impact, relative mass can be taken into account by measuring over a number of exposures.
3. Vector add the two initial momentum vectors to get total initial momentum.
4. Vector add the two final momentum vectors to get total final momentum.

Conservation of momentum is verified if the total initial momentum and total final momentum are equal in both magnitude and direction.

<u>Example:</u>

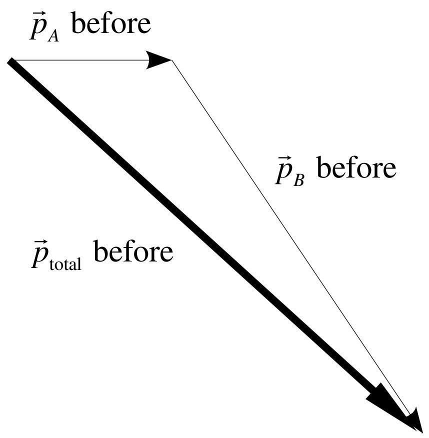
 $m_A = m$  $m_B = 3m$

The multi-image photograph is shown below:

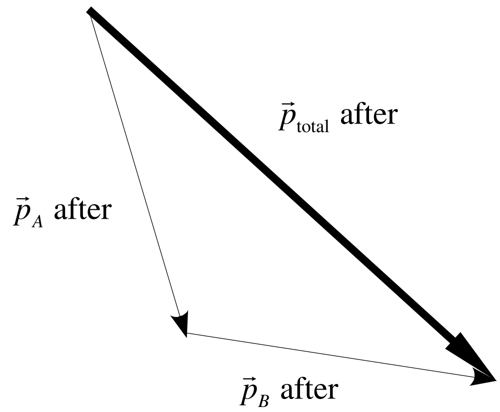


Add the vectors to compare the totals:

Before



After



Notice that the total (resultant) momentum is the same for both before and after. This confirms that momentum is conserved.

Note: This is only perfectly the case in a frictionless environment. If friction exists, some momentum is lost into the surroundings (e.g. air particles).

Conservation of Momentum Problem Solving

1. Draw two diagrams showing the motion of the objects – one before, one after.
2. Make sure all the masses and velocities are shown (as pronumerals e.g. "v_{0_A}" if necessary)
3. Calculate the total momentum before and the total momentum after (in terms of the pronumerals if necessary).
4. Remember that there needs to always be a direction so if necessary use pronumerals for angles. Use scale diagrams or the sine/cosine rules to work with non-right angled shapes.
5. Equate the momentum before with the total momentum after. Rearrange as necessary to find what you're looking for.
6. Always draw vector diagrams (*as well as* the situation diagrams) for 2D problems. In 1D problems, remember that the sign (- or +) represents direction.

Spacecraft propulsion

In order to move around, we take advantage of Newton's third law – we push or pull something so that it pushes or pulls us. A spacecraft has nothing to push against or pull towards, but it can still use Newton's third law and the law of conservation of momentum.

Emission of discrete particles

If a spacecraft emits some kind of fuel, the fuel is experiencing a change in momentum away from the craft. Total momentum must be conserved, so the ship is given an equal amount of momentum in the opposite direction (away from the fuel).

If the fuel is gas, it is often ignited so that it expands and has greater velocity and therefore greater change in momentum, and the craft accelerates faster.

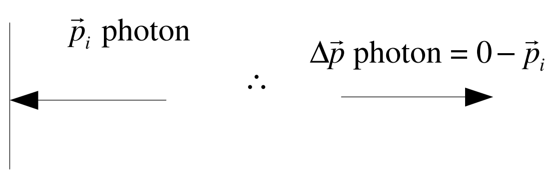
Another form of fuel is ions, which are charged and therefore can be easily accelerated to high velocities using an electric field (see the electric field topics for details on how electric fields accelerate charged particles).

Reflection of light particles (photons)

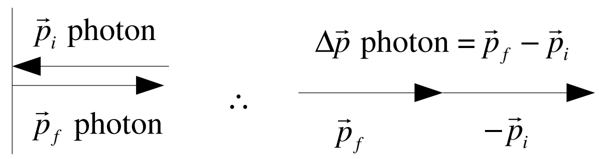
There are other ways of accelerating spacecraft that don't require on-board fuel. One example is that of a solar sail. This theory works on the principle that light 'particles' from the sun (photons) have a small amount of momentum (they have no mass but they interact with matter and have a very high velocity) and that these particles will impart momentum to a spacecraft if they are absorbed or reflected by the solar sail.

To see whether reflection or absorption will impart more force on the spacecraft:

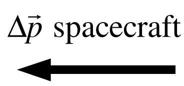
Absorbed (p_f photon = 0)



Reflected

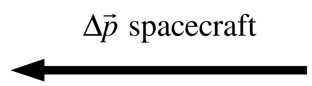


According to conservation of momentum, $\Delta\vec{p}$ spacecraft will be of equal magnitude and in the opposite direction to $\Delta\vec{p}$ photon .



(when photon absorbed)

compared to



(when photon reflected)

A reflected photon's velocity changes more than an absorbed photon, so more force is imparted. Reflection is therefore better for propulsion.