South Australian
Certificate of Education

1

Specialist Mathematics
2020

Question booklet 1

- Questions 1 to 7 (55 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on pages 7 and 16 if you need more space
- Allow approximately 70 minutes
- Approved calculators may be used — complete the box below

Examination information

Materials
- Question booklet 1
- Question booklet 2
- Formula sheet
- SACE registration number label

Instructions
- Show appropriate working and steps of logic in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams

Total time: 130 minutes
Total marks: 100

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Question 1 (8 marks)

Let $f(x)=\dfrac{x^2-1}{x+2}$ and $g(x)=x-2$.

(a) Show that $f(x)=g(x)+\dfrac{3}{x+2}$.

(1 mark)

(b) Figure 1 shows the graph of $g(x)$.

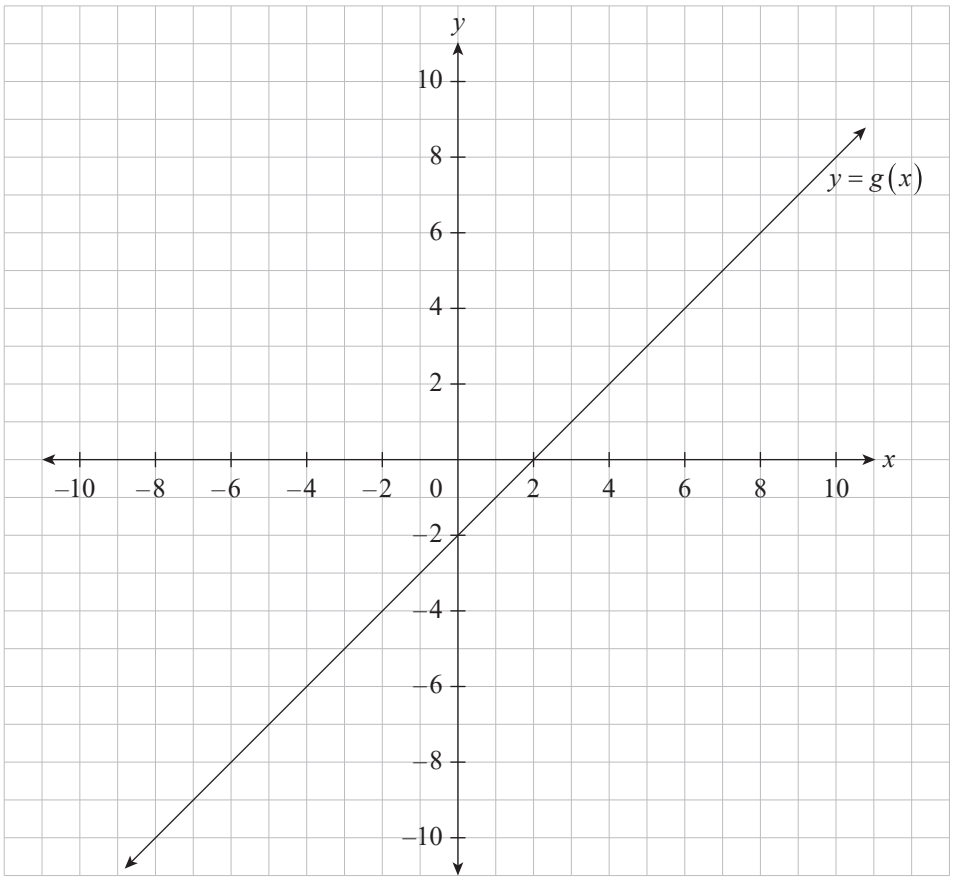

Figure 1

On the axes in Figure 1, sketch and label graphs of each of the functions below, including any asymptotes.

Clearly show the behaviour of the functions near any asymptotes.

(i) $f(x)$ (3 marks)

(ii) $|g(x)|$ (1 mark)

(c) Consider each of the following equations.

State whether solutions exist.

Where solutions exist, state the number of solutions, and find the value(s).

(i) $f(x) = g(x)$

(1 mark)

(ii) $f(x) = |g(x)|$

(2 marks)

Question 2 (8 marks)

Consider the polynomial $P(x) = x^3 + (a-4)x^2 + bx + 5a$, where a and b are real constants.

(a) One zero of $P(x)$ is $2+i$.

(i) State *one* other zero of $P(x)$.

(1 mark)

(ii) Hence find a real quadratic factor of $P(x)$.

(2 marks)

(b) (i) If the remainder is -40 when $P(x)$ is divided by $(x+1)$, show that $6a - b = -35$.

(2 marks)

(ii) One factor of $P(x)$ is $(x-3)$.
Show that $14a + 3b = 9$.

(1 mark)

(iii) Hence find the values of a and b.

(1 mark)

(c) Write the polynomial $P(x)$ as a product of a real linear factor and a real quadratic factor.

(1 mark)

Question 3 (7 marks)

Let $z_1 = (1-i)^m$ and $z_2 = \left(1+i\sqrt{3}\right)^n$, where m and n are positive integers.

(a) Find z_1 and z_2 in $r\,\mathrm{cis}\,\theta$ form.

(3 marks)

(b) (i) If $z_1 = z_2$, show that $m = 2n$.

(1 mark)

(ii) Hence find the smallest positive integers m and n such that $z_1 = z_2$.

(3 marks)

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 2(a)(ii) continued).

Question 4 (9 marks)

The points $A(1, 0, 4)$, $B(5, 4, 0)$, and $C(7, -6, 10)$ form the triangle ABC, as shown in Figure 2.

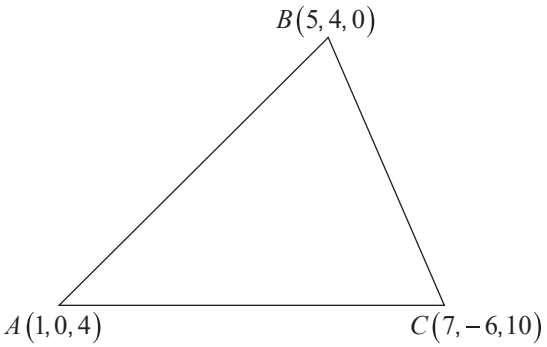

Figure 2

(a) (i) Find $\overrightarrow{AB}$.

(1 mark)

(ii) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(2 marks)

(iii) Find the ***exact*** area of triangle ABC.

(2 marks)

(b) The point $M(4, 3, 1)$ divides AB internally in the ratio $3 : 1$. The point $N(5, -4, 8)$ divides AC internally in the ratio $2 : 1$. That is, $\overrightarrow{AM} = 3\overrightarrow{MB}$ and $\overrightarrow{AN} = 2\overrightarrow{NC}$, as shown in Figure 3.

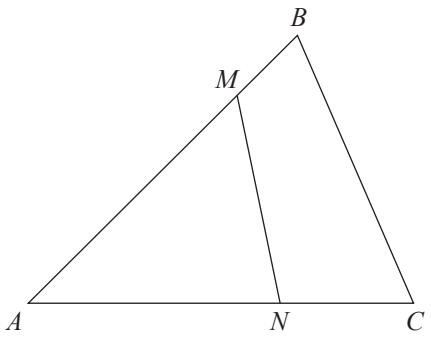

Figure 3

(i) Find $\overrightarrow{AM}$ in terms of $\overrightarrow{AB}$.

(1 mark)

(ii) Find the ***exact*** area of triangle AMN.

(1 mark)

(iii) Find the coordinates of a point, P, on AC, such that:

$$\text{the area of triangle } AMP = \frac{1}{12} \text{ the area of triangle } ABC.$$

(2 marks)

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Question 5 (7 marks)

(a) Use mathematical induction to prove that $7^n + 2$ is divisible by 3 for all positive integers n.

(5 marks)

(b) Hence show that $7^n - 1$ is divisible by 3 for all positive integers n.

(1 mark)

(c) Use parts (a) and (b) to show that $7^{2n} + 7^n - 2$ is divisible by 9 for all positive integers n.

(1 mark)

Question 6 (6 marks)

(a) Use integration by parts to show that

$$\int \arctan x \, \mathrm{d}x = x \arctan x - \frac{1}{2}\ln\left(x^2 + 1\right) + c, \text{ where } c \text{ is a constant.}$$

(2 marks)

(b) Consider the graph of $f\left(x\right) = \sqrt{\arctan x}$ for $x \geq 0$, shown in Figure 4.

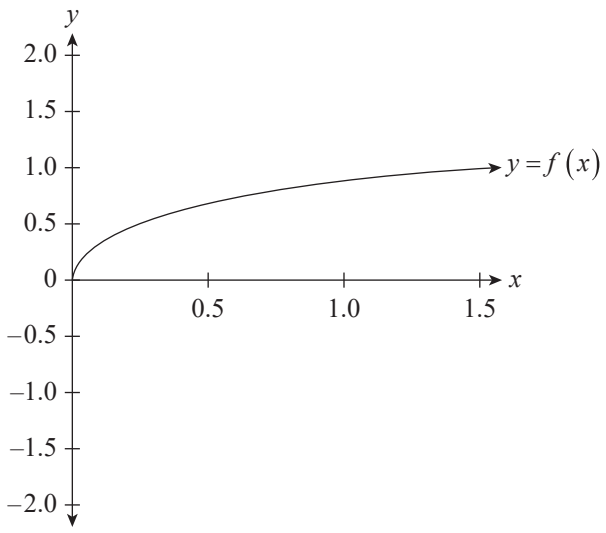

Figure 4

Consider rotating the graph of $f\left(x\right)$ about the x-axis between $x = 0$ and $x = 1$.

(i) Show that the volume of the solid that is obtained by this rotation is given by the equation below.

$$V = \pi \int_0^1 \arctan x \, \mathrm{d}x$$

(1 mark)

(ii) Hence find the ***exact*** volume of this solid.

(3 marks)

Question 7 (10 marks)

(a) Using the fact that $\sin^3 x = \sin^2 x \sin x$ show that

$$\int \sin^3 x \, dx = -\cos x + \frac{1}{3}\cos^3 x + c, \text{ where } c \text{ is a constant.}$$

(3 marks)

(b) Figure 5 shows the graph of $f(x) = \sin^3 x$ for $0 \leq x \leq \frac{\pi}{2}$ and the graph of $y = x$ for $x \geq 0$.

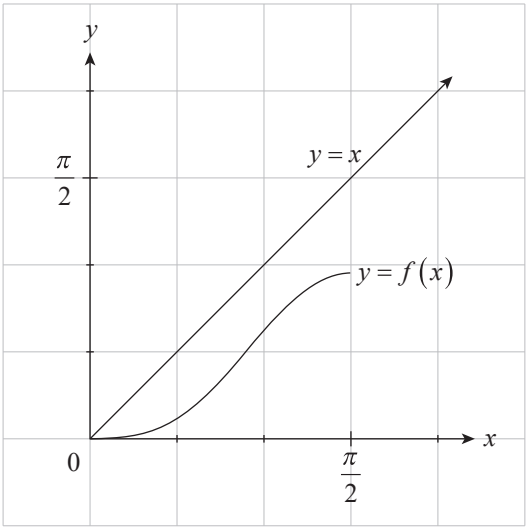

Figure 5

(i) Find $f\left(\frac{\pi}{2}\right)$.

(1 mark)

(ii) Explain why the function $f(x)$ has an inverse function.

(1 mark)

(iii) On Figure 5, draw the graph of the inverse function, $f^{-1}(x)$, using symmetry about the line $y = x$. (1 mark)

(c) Use parts (a) and (b) to show that $\int_0^1 f^{-1}(x)\,\mathrm{d}x = \frac{\pi}{2} - \frac{2}{3}$.

(4 marks)

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 4(b)(iii) continued).

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Question booklet 2

- Questions 8 to 10 (45 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on pages 5 and 12 if you need more space
- Allow approximately 60 minutes
- Approved calculators may be used — complete the box below

2

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Question 8 (15 marks)

(a) Consider the planes P_1 and P_2 that are defined by the equations below.

$$P_1 : \ x + y + 2z = 4$$
$$P_2 : \ 2x - y + z = 8$$

(i) Show that P_1 and P_2 intersect at line l_1, which has the following parametric equations:

$$\begin{cases} x = 4 - t \\ y = -t \\ z = t \end{cases}$$ where t is a real parameter.

Clearly state all row operations.

(3 marks)

(ii) Show that the points $A(0, -4, 4)$ and $B(1, -3, 3)$ are on l_1.

(2 marks)

(iii) Show that the point $C(4, 2, 2)$ is on P_2.

(1 mark)

(b) Figure 6 shows P_1 and P_2, and the line l_1 where P_1 and P_2 intersect.
The normal to P_2 through C meets P_1 at the point D.

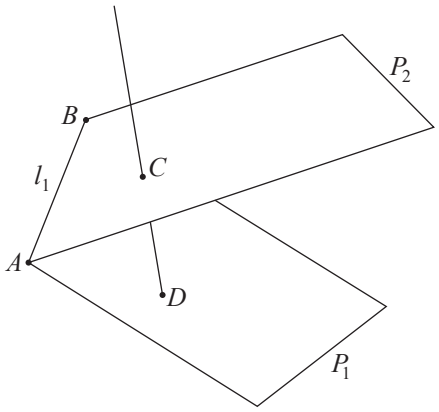


Figure 6

(i) Find the equation of the normal to P_2 through C.

(2 marks)

(ii) Show that D has coordinates $(0, 4, 0)$.

(2 marks)

(iii) From part (a)(i), the parametric equations for l_1 are:

$$\begin{cases} x = 4 - t \\ y = -t \\ z = t \end{cases}$$ where t is a real parameter.

Find the coordinates of the point on l_1 that is closest to $D(0, 4, 0)$.

(3 marks)

(iv) How much closer is D to P_2 than it is to l_1?

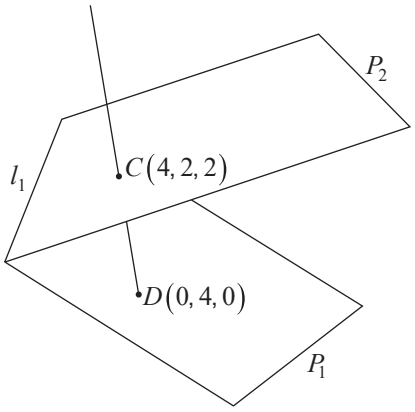


Figure 7

(2 marks)

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 8(a)(ii) continued).

Question 9 (16 marks)

(a) Consider the parametric curve defined by the following equations:

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t + 2 \end{cases} \quad \text{where } 0 \le t < 2\pi.$$

(i) Show that the points on the curve satisfy the Cartesian equation $x^2 + (y-2)^2 = 1$.

(1 mark)

(ii) On the axes in Figure 8, draw the curve described in part (a)(i).

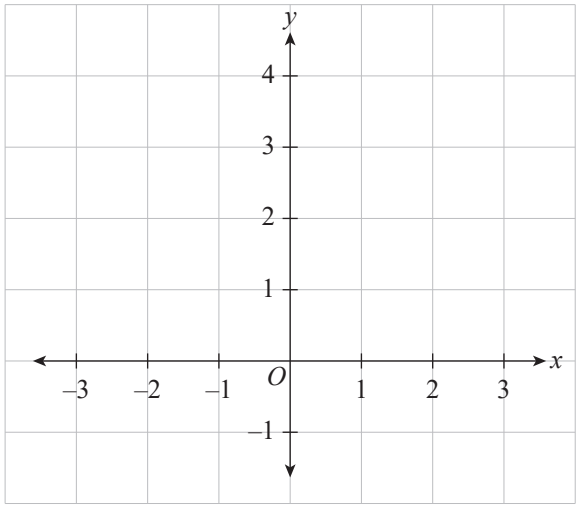


Figure 8

(2 marks)

(iii) Using part (a)(i) and implicit differentiation, show that $\frac{dy}{dx} = \frac{x}{2-y}$.

(2 marks)

(iv) On the curve that you drew on Figure 8, mark the point A for which $t = \frac{\pi}{6}$. (1 mark)

(v) Find the equation, in ***exact*** form, of the tangent to the curve at A.

(2 marks)

(b) Figure 9 shows a circle in the complex plane and the tangent to the circle from the origin O, meeting the circle at the point P in the first quadrant.

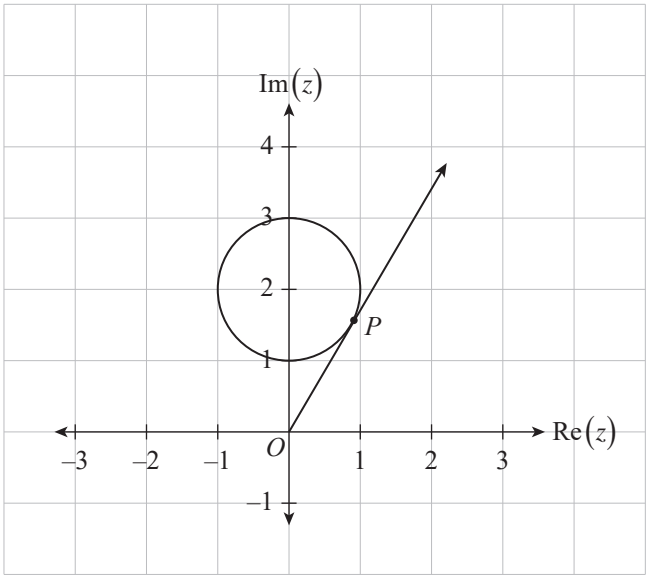


Figure 9

(i) Write an equation in z that describes the complex numbers that are represented by the circle.

(2 marks)

(ii) Find the complex number represented by P in ***exact*** polar form.

(2 marks)

(iii) Hence explain why, for all z on the circle, $\frac{\pi}{3} \leq \arg z \leq \frac{2\pi}{3}$.

(1 mark)

(iv) Find the ***exact*** coordinates of P.

(1 mark)

(c) Transfer point A from Figure 8 (on page 6) to Figure 10 below.

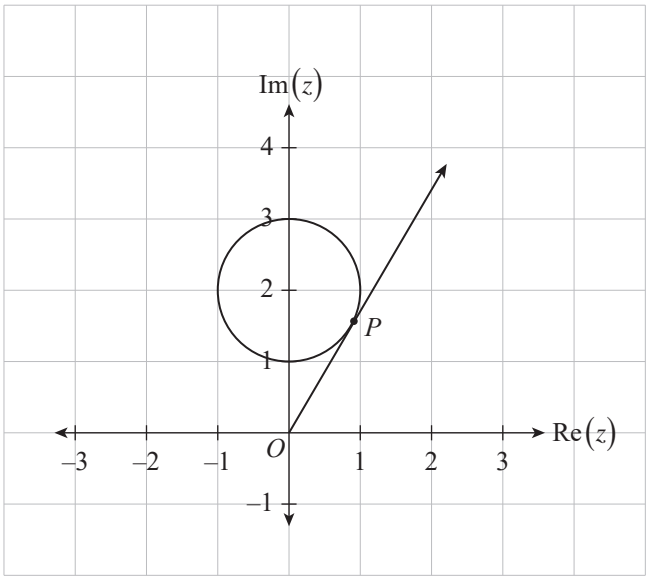


Figure 10

Find the ***exact*** coordinates of the point X, where the tangents to the circle at A and at P intersect.

(2 marks)

Question 10 (14 marks)

Figure 11 shows the slope field for the differential equation $\frac{dy}{dx} = 4 - y^2$.

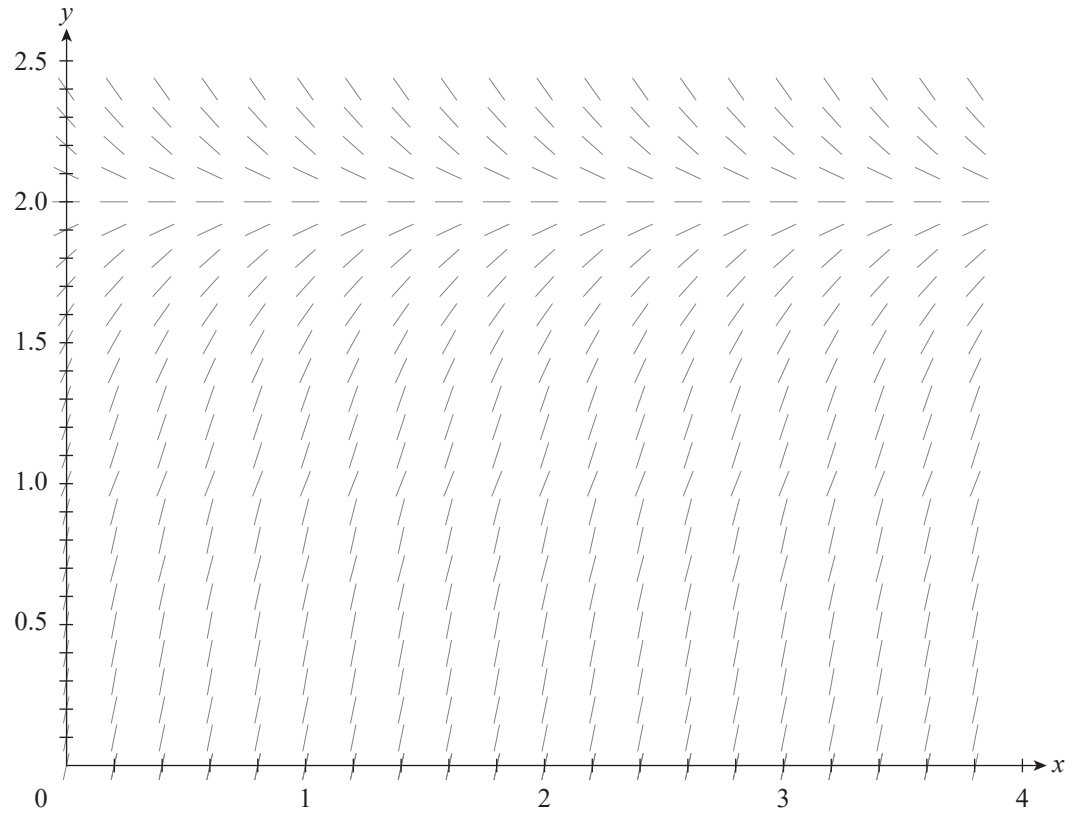


Figure 11

(a) On Figure 11, draw the solution curve for the differential equation, starting at $x = 1,\ y = 0$.

(3 marks)

Question 10 continues on page 10.

(b) Show that $\frac{4}{4-y^2} = \frac{1}{2+y} + \frac{1}{2-y}$.

(1 mark)

(c) Hence use integration techniques to solve the differential equation $\frac{dy}{dx} = 4 - y^2$ when $y(1) = 0$ to show that $y = 2\left(\frac{e^{4(x-1)} - 1}{1 + e^{4(x-1)}}\right)$.

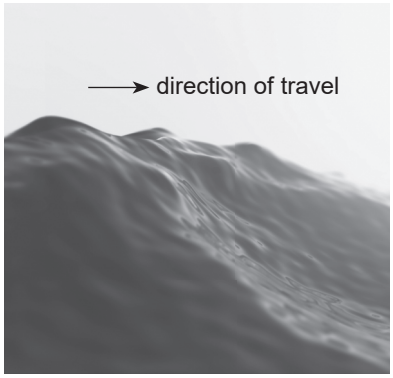
(6 marks)

(d) The height $h(x)$, in metres, of a wave in the ocean can be modelled by the following equation:

$$h(x) = 4 - y^2$$

where x is the distance that the wave has travelled towards the beach, in metres, when $x \geq 1$, and

$y = 2\left(\dfrac{e^{4(x-1)} - 1}{1 + e^{4(x-1)}}\right)$ is the function from part (c) defined for $x \geq 1$.



Source: © Rostislav Zatonskiy | Dreamstime.com

(i) On the axes in Figure 12, draw the graph of $h(x) = 4 - y^2$, for $x \geq 1$.

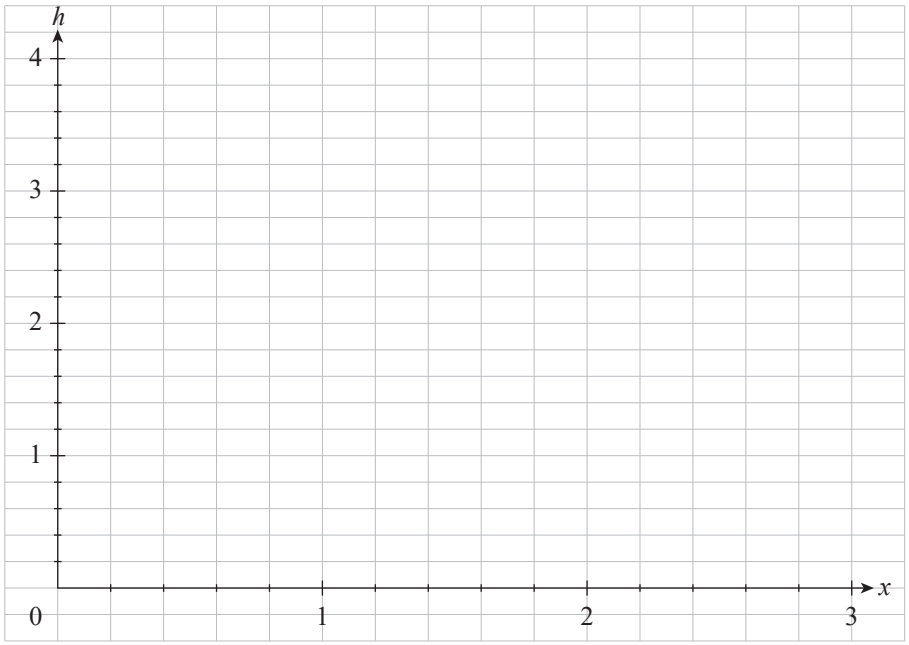


Figure 12

(3 marks)

(ii) Find the value of x for which the wave height is decreasing at the greatest rate.

(1 mark)

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 9(a)(iii) continued).

SPECIALIST MATHEMATICS FORMULA SHEET

Circular functions

$$\sin^2 A + \cos^2 A = 1$$

$$\tan^2 A + 1 = \sec^2 A$$

$$1 + \cot^2 A = \operatorname{cosec}^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

$$\sin A \pm \sin B = 2 \sin \tfrac{1}{2}(A \pm B) \cos \tfrac{1}{2}(A \mp B)$$

$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B)$$

$$\cos A - \cos B = -2 \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B)$$

Matrices and determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = |A| = ad - bc$ and

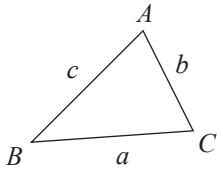
$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Measurement

Area of sector, $A = \frac{1}{2} r^2 \theta$, where θ is in radians.

Arc length, $l = r\theta$, where θ is in radians.

In any triangle ABC:



Area of triangle $= \frac{1}{2} ab \sin C$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Quadratic equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Distance from a point to a plane

The distance from (x_1, y_1, z_1) to $Ax + By + Cz + D = 0$ is given by

$$\frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Derivatives

$f(x) = y$	$f'(x) = \dfrac{\mathrm{d}y}{\mathrm{d}x}$
$\arcsin x$	$\dfrac{1}{\sqrt{1 - x^2}}$
$\arccos x$	$\dfrac{-1}{\sqrt{1 - x^2}}$
$\arctan x$	$\dfrac{1}{1 + x^2}$

Properties of derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)\,g(x)\big) = f'(x)g(x) + f(x)g'(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\big[g(x)\big]^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$$

Arc length along a parametric curve

$$l = \int_a^b \sqrt{\boldsymbol{v} \bullet \boldsymbol{v}}\, \mathrm{d}t, \text{ where } a \le t \le b.$$

Integration by parts

$$\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x$$

Volumes of revolution

About x axis, $V = \displaystyle\int_a^b \pi y^2 \mathrm{d}x$, where y is a function of x.

About y axis, $V = \displaystyle\int_c^d \pi x^2 \mathrm{d}y$, where y is a one-to-one function of x.