South Australian
Certificate of Education

1

Specialist Mathematics
2019

Question booklet 1

Part 1 (Questions 1 to 10) 75 marks
- Answer ***all*** questions in Part 1
- Write your answers in this question booklet
- You may write on page 22 if you need more space
- Allow approximately 95 minutes
- Approved calculators may be used — complete the box below

Examination information

Materials
- Question booklet 1 (Part 1)
- Question booklet 2 (Part 2)
- Formula sheet
- SACE registration number label

Instructions
- Show appropriate working and steps of logic in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams and graphical representations

Total time: 190 minutes
Total marks: 150

Attach your SACE registration number label here

Graphics calculator

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PART 1 (Questions 1 to 10)

(75 marks)

Question 1 (5 marks)

(a) Show that $\frac{4}{x^2-4}=\frac{1}{x-2}-\frac{1}{x+2}$.

(1 mark)

(b) (i) Hence show that $\int \frac{1}{x^2-4}\,\mathrm{d}x=\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+c$.

(3 marks)

(ii) Find the exact value of $\int_0^1 \frac{1}{x^2 - 4} \mathrm{d}x$.

(1 mark)

Question 2 (9 marks)

A curve has the following parametric equations:

$$\begin{cases} x(t) = \cos 2t \\ y(t) = \sqrt{\sin t} \end{cases} \quad \text{where } 0 < t \leq \frac{\pi}{2}.$$

(a) On the axes in Figure 1, draw a graph of the curve defined by these parametric equations.

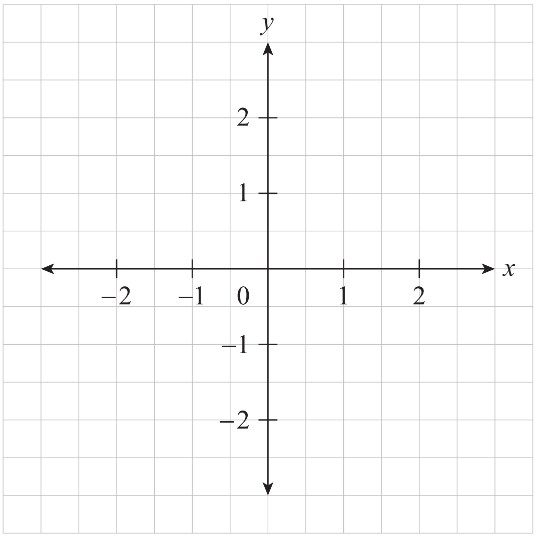

Figure 1

(3 marks)

(b) Using $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'(t)}{x'(t)}$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{8y^3}$.

(3 marks)

(c) Using part (b), find the exact slope of the tangent to the curve at $x = \frac{1}{2}$.

(3 marks)

Question 3 (8 marks)

Figure 2 shows the quadrilateral $ABCD$, where $\overrightarrow{AB} = \boldsymbol{a}$, $\overrightarrow{BC} = \boldsymbol{b}$, and $\overrightarrow{CD} = \boldsymbol{c}$.

The points E, F, G, and H are the midpoints of the sides AB, BC, CD, and DA respectively.

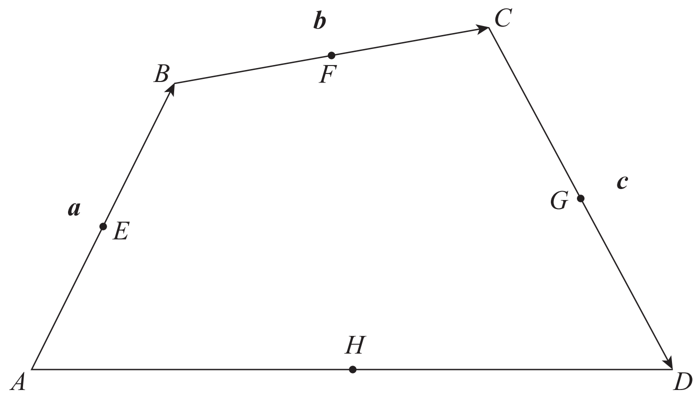

Figure 2

(a) Find the following vectors in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

(i) $\overrightarrow{AD}$

(1 mark)

(ii) $\overrightarrow{EF}$

(1 mark)

(iii) $\overrightarrow{HG}$

(2 marks)

(b) (i) Explain why $EFGH$ is a parallelogram.

(2 marks)

(ii) Show that the area of $EFGH$ is $\frac{1}{4}\left|(\boldsymbol{a}\times\boldsymbol{b})+(\boldsymbol{a}\times\boldsymbol{c})+(\boldsymbol{b}\times\boldsymbol{c})\right|$.

(2 marks)

Question 4 (7 marks)

Consider the function $f\left(x\right)=\dfrac{1}{x\left(x-2\right)}$.

(a) (i) On the axes in Figure 3, sketch the graph of $y=f\left(x\right)$.

Clearly show the behaviour of the function near the asymptote(s).

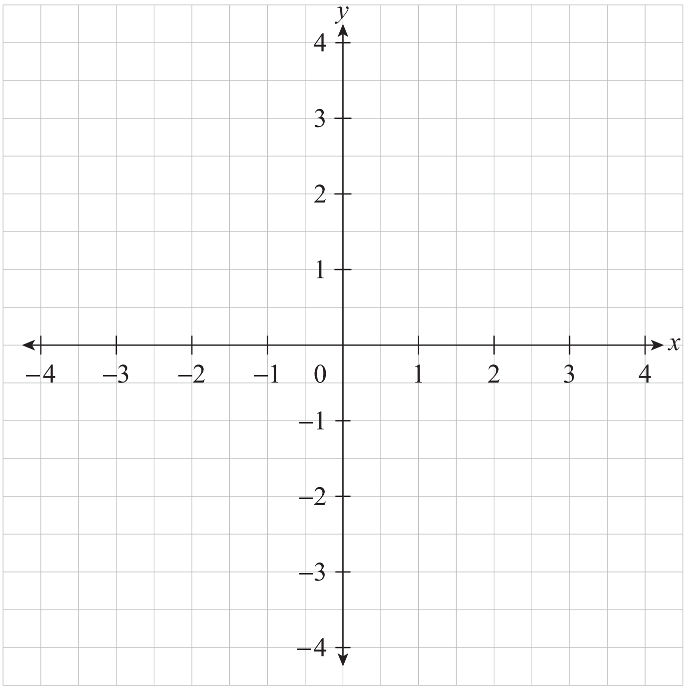

Figure 3

(3 marks)

(ii) On the axes in Figure 4, sketch the graph of $y = \left|f(x)\right|$.

Clearly show the behaviour of the function near the asymptote(s).

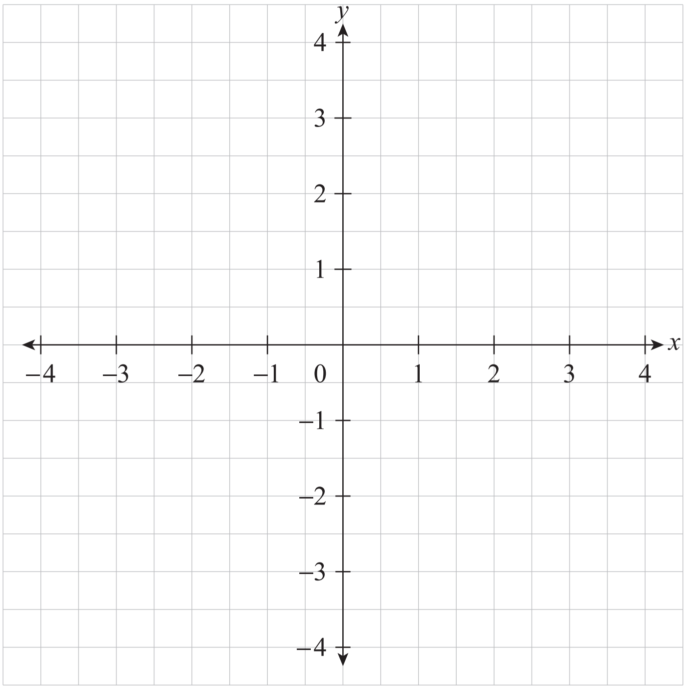

Figure 4

(1 mark)

(b) On the axes in Figure 5, sketch the graph of $y = f\left(|x|\right)$.

Clearly show the behaviour of the function near the asymptote(s).

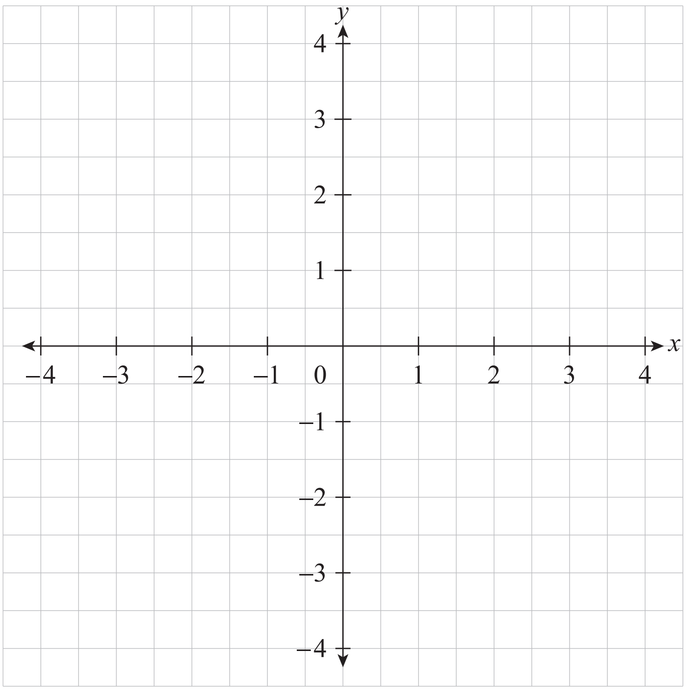

Figure 5

(3 marks)

Question 5 (8 marks)

(a) Use mathematical induction to prove that

$$4+4^2+4^3+\ldots+4^n=\frac{4}{3}\left(4^n-1\right)$$

where n is a positive integer.

(6 marks)

(b) Hence show that $3+15+63+\ldots+16\,777\,215=22\,369\,608$.

(2 marks)

Question 6 (9 marks)

(a) Consider the planes that are defined by the following system of equations:

$$P_1 : 3x - y + 2z = 7$$
$$P_2 : \ 2x - y - z = 12.$$

(i) Write this system of equations as an augmented matrix.

(1 mark)

(ii) Clearly stating all row operations, show that there are infinite solutions to this system of equations, and give the solutions in parametric form.

(3 marks)

(iii) Interpret your answer to part (a)(ii) geometrically.

(1 mark)

(b) A third plane is added to the system of equations:

$$P_3 : x - y - (k+2)z = 17.$$

For the system of three equations:

(i) find the value of k for which there are infinite solutions.

(2 marks)

(ii) find the solution for all other values of k.

(1 mark)

(iii) interpret your answer to part (b)(ii) geometrically.

(1 mark)

Question 7 (7 marks)

Consider the complex numbers $z = i$ and $w = \dfrac{\sqrt{2}}{1-i}$.

(a) (i) Express z in exact polar form.

(1 mark)

(ii) Express w in exact polar form.

(1 mark)

(b) On the Argand diagram in Figure 6, draw and label z, w, and $z + w$.

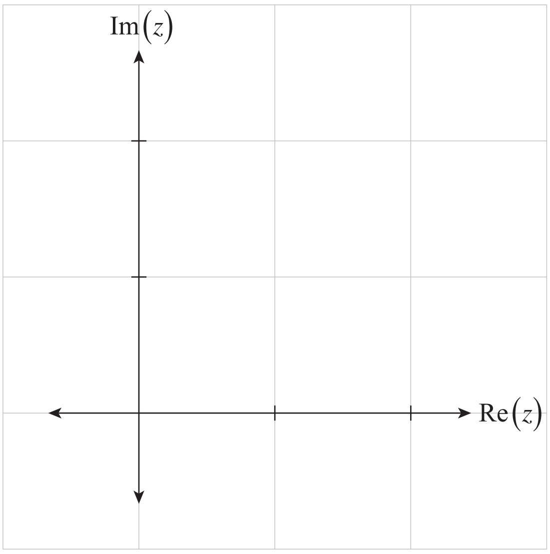

Figure 6

(2 marks)

(c) (i) Using z, w, and $z+w$ from part (b), show that $\arg(z+w)=\dfrac{3\pi}{8}$.

(1 mark)

(ii) Write $z+w$ in Cartesian form.

(1 mark)

(iii) Hence show that $\tan\dfrac{3\pi}{8}=1+\sqrt{2}$.

(1 mark)

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Question 8 (6 marks)

The flight path of a fly is given by solutions to the differential equation below.

$$\frac{dy}{dx} = \frac{x\left(x^2 - 1\right)}{y\left(2y^2 - 1\right)}$$

(a) On the slope field in Figure 7, draw the flight path that passes through the origin $(0, 0)$.

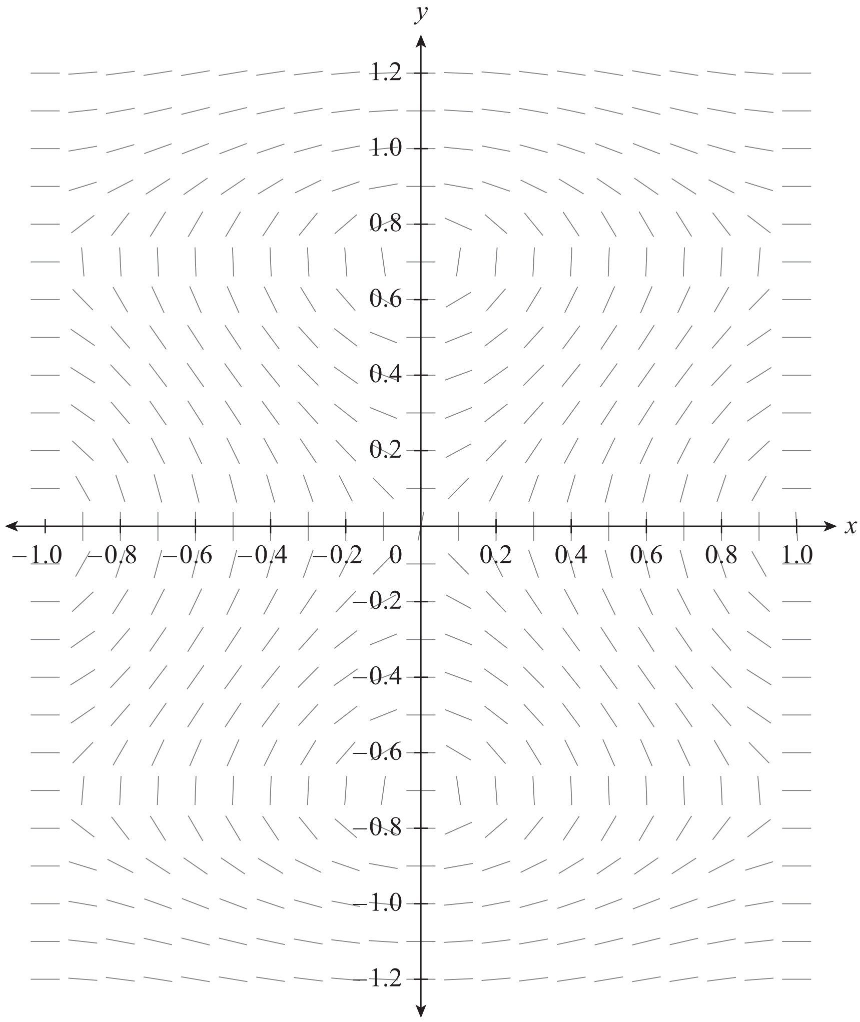

Figure 7 (3 marks)

(b) The differential equation can be solved by finding the solution to the integral equation below.

$$\int y\left(2y^2-1\right)\mathrm{d}y=\int x\left(x^2-1\right)\mathrm{d}x$$

Find the equation of the solution curve that passes through the origin $(0,0)$.

(3 marks)

Question 9 (7 marks)

$P(x)$ is a real cubic polynomial. When $P(x)$ is divided by $(x-1)$, the remainder is 35, and when it is divided by $(x+2)$, the remainder is 80.

(a) Find the values of a and b if $P(x)=Q(x)\left(x^2+x-2\right)+(ax+b)$.

(3 marks)

(b) (i) If $(x-2)$ is a factor of $P(x)$, show that $Q(2)=-5$.

(1 mark)

(ii) If the leading coefficient of $P(x)$ is 1, show that $Q(x)=x-7$.

(2 marks)

(iii) Hence find the expanded form of $P(x)$.

(1 mark)

Question 10 (9 marks)

Scientists can study the growth of the world's population using mathematical modelling.

For one model, the rate of population growth since the year 2000 is given by the differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{0.002N(12.5 - N)}{12.5}$$

where N is the population of the world (in billions) at time t (in years since 2000).

(a) (i) Show that $\dfrac{12.5}{N(12.5 - N)} = \dfrac{1}{N} + \dfrac{1}{12.5 - N}$.

(1 mark)

(ii) Hence solve the differential equation with the initial condition $t = 0$, $N = 6.08$ to show that

$$N = \frac{12.5}{1 + 1.06e^{-0.002t}}.$$

(5 marks)

(b) According to this model, in what year is the population growing at its greatest rate?

(2 marks)

(c) According to this model, what is the limiting value of the world's population?

(1 mark)

You may write on this page if you need more space to finish your answers to questions in Part 1. Make sure to label each answer carefully (e.g. 6(b)(i) continued).

South Australian Certificate of Education

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Question booklet 2

Part 2 (Questions 11 to 15) 75 marks
- Answer ***all*** questions in Part 2
- Write your answers in this question booklet
- You may write on pages 9 and 19 if you need more space
- Allow approximately 95 minutes
- Approved calculators may be used — complete the box below

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Copy the information from your SACE label here

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PART 2 (Questions 11 to 15)

(75 marks)

Question 11 (15 marks)

(a) Calculate the vector (cross) product $[1,-1,-1]\times[1,0,1]$.

(1 mark)

(b) Consider the planes P_1 and P_2 that are defined by the following equations:

$$P_1: x-y-z=4$$
$$P_2: \quad x+z=9.$$

Figure 8 shows P_1, P_2, and the line l_1, where P_1 and P_2 intersect.

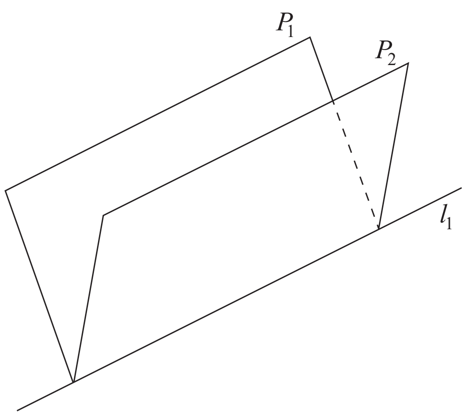

Figure 8

(i) Show that the point $X(9,5,0)$ is on both planes.

(1 mark)

(ii) Hence or otherwise, show that l_1 has the following parametric equations:

$$\begin{cases} x = 9 - t \\ y = 5 - 2t \\ z = t \end{cases} \quad \text{where } t \text{ is real.}$$

(2 marks)

(c) Consider the line l_2, which has the following parametric equations:

$$\begin{cases} x = 3 + 3s \\ y = -s \\ z = 3 \end{cases} \quad \text{where } s \text{ is real.}$$

(i) (1) Show that l_2 intersects l_1.

(2 marks)

(2) Find Y, the point where l_1 and l_2 intersect.

(1 mark)

The line l_2 lies on the plane P_3.

Plane P_3 intersects P_1 and P_2 along the common line l_1, as shown in Figure 9.

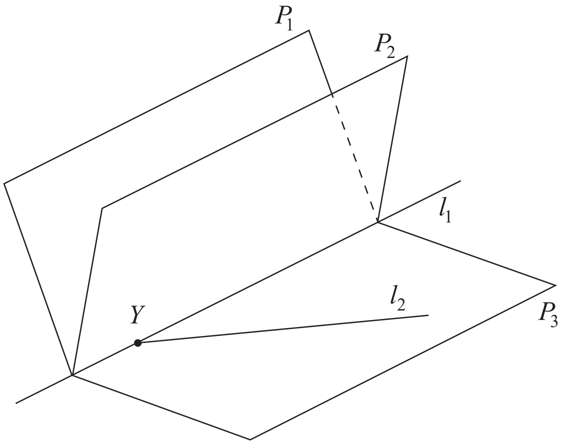

Figure 9

(ii) Show that the equation of P_3 is $x + 3y + 7z = 24$.

(3 marks)

(d) The line l_3 is parallel to l_2, as shown in Figure 10.

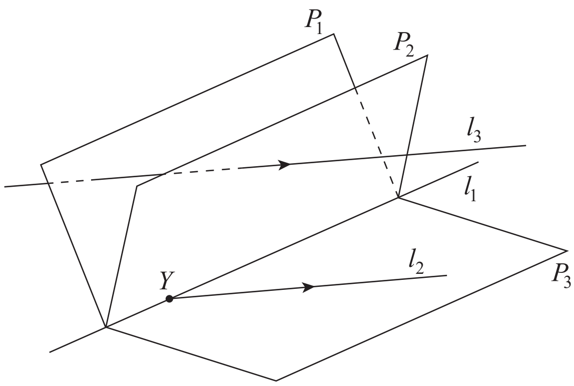

Figure 10

(i) Line l_3 passes through the origin.

Write an equation for l_3.

(1 mark)

(ii) Verify that l_3 does ***not*** lie on P_3.

(1 mark)

(e) Particles are fired from a source located at the origin and travel along l_3.

(i) If the particles travel at a constant speed of $\sqrt{10}$ units/second, show that the particles pass through P_1, 1 second after they have been fired.

(2 marks)

(ii) How many more seconds elapse before the particles pass through P_2?

(1 mark)

Question 12 (15 marks)

The following parametric equations describe the motion of a particle moving in a spiral pattern towards the origin:

$$\begin{cases} x(t) = e^{-\frac{1}{5}t}\sin t \\ y(t) = e^{-\frac{1}{5}t}\cos t \end{cases} \quad \text{for } 0 \le t \le 3\pi$$

where t represents time in seconds, and x and y are distances measured in centimetres.

(a) On the axes in Figure 11, sketch the curve defined by these parametric equations.

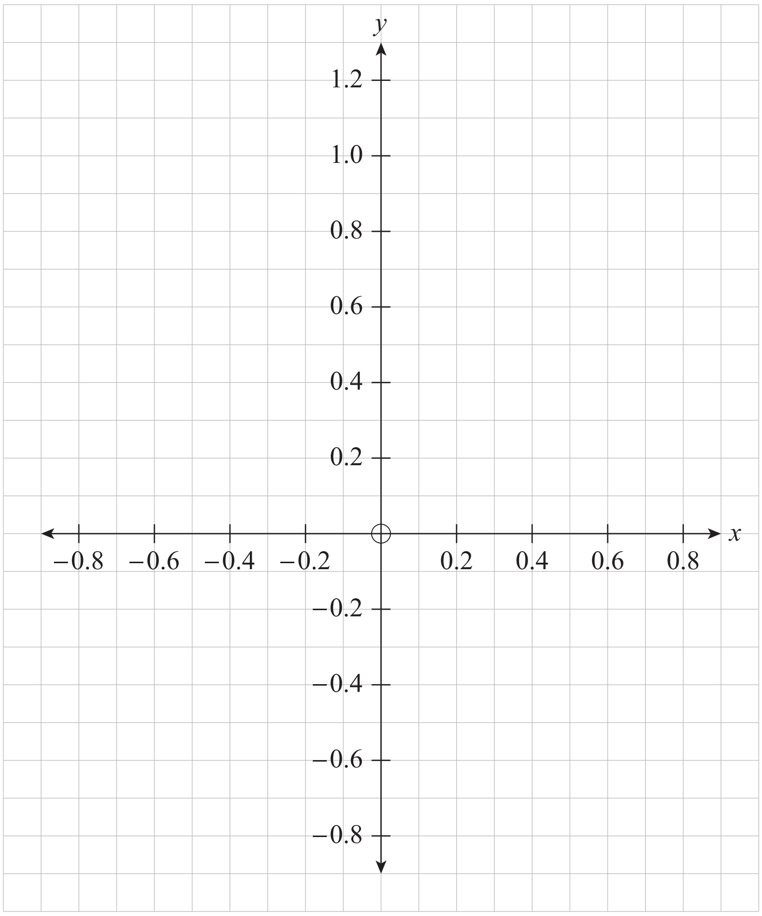

Figure 11 (3 marks)

(b) Show that the velocity vector of the particle is

$$\boldsymbol{v} = -\frac{1}{5}e^{-\frac{1}{5}t}\left[(\sin t - 5\cos t), (\cos t + 5\sin t)\right].$$

(3 marks)

(c) Find the velocity vector of the particle at $t = \frac{\pi}{2}$.

(2 marks)

(d) Find the speed of the particle at $t = \frac{\pi}{2}$.

(1 mark)

(e) (i) Write an expression in terms of t for finding the length of the path that the particle has taken for $0 \le t \le \pi$.

(2 marks)

(ii) Show that this expression can be simplified to $\frac{\sqrt{26}}{5}\int_0^{\pi} e^{-\frac{1}{5}t}\,\mathrm{d}t$.

(2 marks)

(iii) Use this simplified expression to find the length of the path for $0 \le t \le \pi$, correct to three significant figures.

(2 marks)

You may write on this page if you need more space to finish your answers to questions in Part 2.
Make sure to label each answer carefully (e.g. 11(d)(ii) continued).

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Question 13 (15 marks)

Figure 12 shows the graph of $g(x) = \arcsin(x^2)$ for $-1 \le x \le 1$.

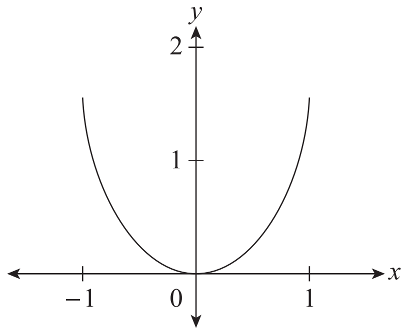

Figure 12

(a) (i) Explain why $g(x)$ does not have an inverse.

(1 mark)

(ii) Explain why $f(x) = \arcsin(x^2)$ for $0 \le x \le 1$ does have an inverse.

(1 mark)

(b) Figure 13 shows the graph of $y = f(x)$.

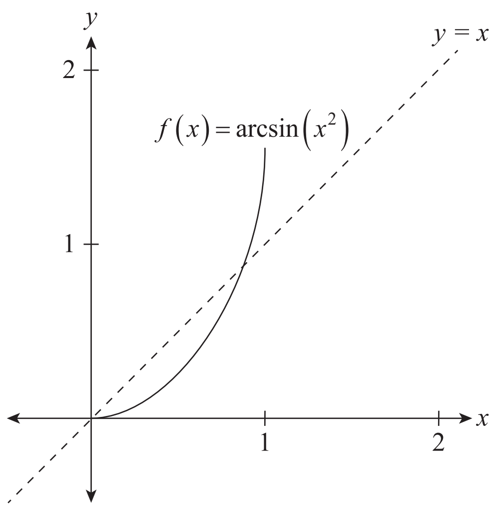

Figure 13

(i) On the axes in Figure 13, sketch the graph of $f^{-1}(x)$. (2 marks)

(ii) Write the exact domain of $f^{-1}(x)$.

(1 mark)

(iii) Find the equation of $f^{-1}(x)$.

(2 marks)

(c) If $y = \arcsin\left(x^2\right)$, then $x^2 = \sin y$.

Hence use implicit differentiation to show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{\sqrt{1-x^4}}$.

(3 marks)

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(d) (i) Use integration by parts to show that

$$\int x\arcsin\left(x^2\right)\mathrm{d}x = \frac{1}{2}x^2\arcsin\left(x^2\right) + \frac{1}{2}\sqrt{1-x^4} + c$$

where c is a constant.

(3 marks)

(ii) Hence find the exact value of $\int_0^1 x\arcsin\left(x^2\right)\mathrm{d}x$.

(2 marks)

Question 14 (15 marks)

The complex number w is shown on the Argand diagram in Figure 14.

The dashed rays represent the complex numbers such that $\arg z = \frac{\pi}{6}$, $\arg z = \frac{\pi}{4}$, and $\arg z = \frac{\pi}{3}$.

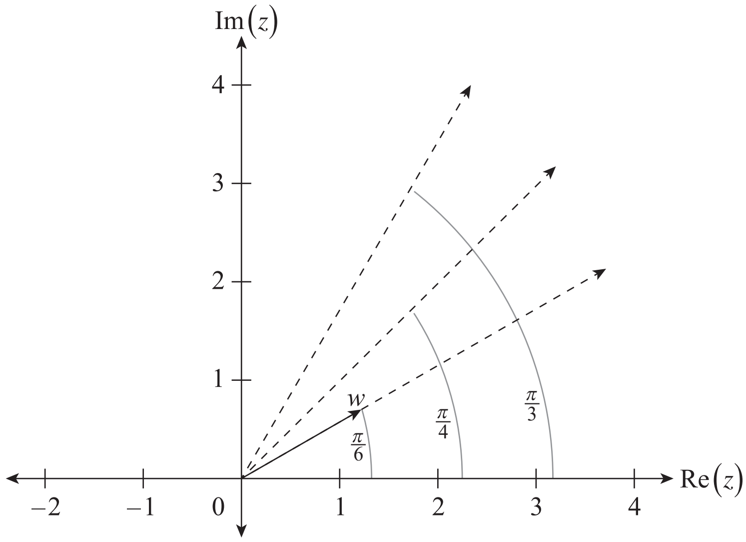


Figure 14

(a) On the Argand diagram in Figure 14:

(i) draw the approximate position of w^3. (1 mark)

(ii) sketch the set of complex numbers z such that $\left|z - w^3\right| = \left|w\right|$. (2 marks)

(b) (i) Write $z = \frac{\sqrt{6}}{2} + \frac{i\sqrt{2}}{2}$ in exact polar form.

(1 mark)

(ii) Use de Moivre's theorem to show that $z^6 = -8$.

(1 mark)

(c) (i) Show that the solutions of $z^6 = -8$ are: $z_1 = \sqrt{2}\text{cis}\left(\frac{\pi}{6}\right)$; $z_2 = \sqrt{2}\text{cis}\left(\frac{\pi}{2}\right)$; $z_3 = \sqrt{2}\text{cis}\left(\frac{5\pi}{6}\right)$;

$z_4 = \sqrt{2}\text{cis}\left(-\frac{5\pi}{6}\right)$; $z_5 = \sqrt{2}\text{cis}\left(-\frac{\pi}{2}\right)$; and $z_6 = \sqrt{2}\text{cis}\left(-\frac{\pi}{6}\right)$.

(2 marks)

On the Argand diagram in Figure 15, P represents ${z_1}^3$ and Q represents $2z_1$, where $z_1 = \sqrt{2}\text{cis}\left(\frac{\pi}{6}\right)$.

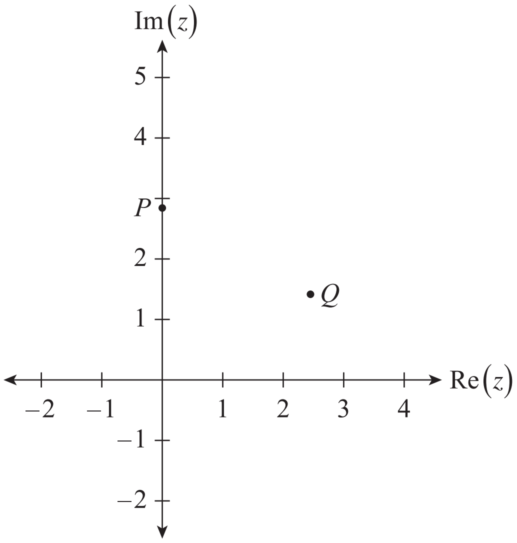


Figure 15

(ii) On the Argand diagram in Figure 15:

(1) sketch $|z| = \sqrt{2}$. (1 mark)

(2) draw and label z_1, z_2, z_3, z_4, z_5, and z_6. (2 marks)

(iii) Find the exact value of $\left| z_1^3 - 2z_1 \right|$.

(2 marks)

Circles of radius $\sqrt{2}$ are drawn with centres P and Q.

(iv) Deduce that these circles touch.

(1 mark)

(v) Find the point where the circles touch, and write it in exact $x + iy$ form.

(2 marks)

Question 15 (15 marks)

(a) Using integration by parts, show that $\int \ln x \, dx = x \ln x - x + c$, where c is a constant, for $x > 0$.

(2 marks)

(b) Consider the function $y = e^{x^2} - 15$, where $y \geq 0$, as shown in Figure 16.

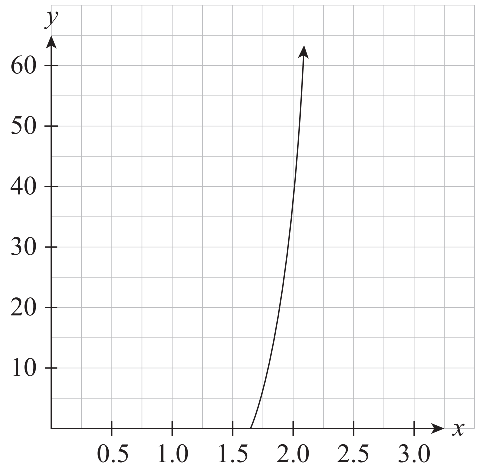


Figure 16

(i) This curve, y, is rotated about the y-axis bounded by the lines $y = 0$ and $y = h$ (where $h > 0$), forming a solid with volume V.

Show that $V = \pi\left[(h+15)(\ln(h+15)-1)-15(\ln 15-1)\right]$.

(3 marks)

The volume found in part (b)(i) represents the volume of a container that holds water.

The container, initially full of water, begins leaking from a small hole in the base of the container. The depth of water in the container, h cm, varies with time t, measured in seconds.

(ii) Show that the rate at which water leaks from the container is given by

$$\frac{dV}{dt} = \frac{dh}{dt}\pi \ln(h+15).$$

(3 marks)

(c) Another formula for the rate at which the container leaks is $\frac{dV}{dt} = -0.98\sqrt{h}$.

(i) Using part (b)(ii), show that $\int 0.98 dt = -\pi \int \frac{\ln(h+15)}{\sqrt{h}} dh$.

(2 marks)

(ii) Suppose the initial depth of the water in the container is 35 cm.

Using the result from part (c)(i), explain clearly why the time taken for the container to empty, t_e, is

$$t_e = \frac{\pi}{0.98}\int_0^{35} \frac{\ln(h+15)}{\sqrt{h}}\,dh.$$

(2 marks)

(iii) Hence find the time taken for the container to empty, correct to the nearest second.

(1 mark)

(d) Now consider the situation where the container is initially empty. Water is pumped into the container at a constant rate of 2.43 cm^3 per second.

At the same time, water leaks from the container at a rate given by $\frac{dV}{dt} = -0.98\sqrt{h}$.

Determine the depth at which the level of water in the container is not increasing and not decreasing.

(2 marks)

You may write on this page if you need more space to finish your answers to questions in Part 2.
Make sure to label each answer carefully (e.g. 15(b)(i) continued).

SPECIALIST MATHEMATICS FORMULA SHEET

Circular functions

$\sin^2 A + \cos^2 A = 1$

$\tan^2 A + 1 = \sec^2 A$

$1 + \cot^2 A = \operatorname{cosec}^2 A$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2\sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$\quad = 2\cos^2 A - 1$

$\quad = 1 - 2\sin^2 A$

$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$\sin A \pm \sin B = 2\sin\tfrac{1}{2}(A \pm B)\cos\tfrac{1}{2}(A \mp B)$

$\cos A + \cos B = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$

$\cos A - \cos B = -2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$

Matrices and determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = |A| = ad - bc$ and

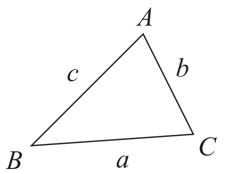
$A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Measurement

Area of sector, $A = \frac{1}{2}r^2\theta$, where θ is in radians.

Arc length, $l = r\theta$, where θ is in radians.

In any triangle ABC:



Area of triangle $= \frac{1}{2}ab\sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$a^2 = b^2 + c^2 - 2bc\cos A$

Quadratic equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Distance from a point to a plane

The distance from (x_1, y_1, z_1) to

$Ax + By + Cz + D = 0$ is given by

$\dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

Derivatives

$f(x) = y$	$f'(x) = \dfrac{dy}{dx}$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\dfrac{-1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$

Properties of derivatives

$\dfrac{d}{dx}\big(f(x)\,g(x)\big) = f'(x)g(x) + f(x)g'(x)$

$\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

$\dfrac{d}{dx}f(g(x)) = f'(g(x))\,g'(x)$

Arc length along a parametric curve

$l = \displaystyle\int_a^b \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}\,dt$, where $a \le t \le b$.

Integration by parts

$\displaystyle\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$

Volumes of revolution

About x axis, $V = \displaystyle\int_a^b \pi y^2\,dx$, where y is a function of x.

About y axis, $V = \displaystyle\int_c^d \pi x^2\,dy$, where y is a one-to-one function of x.