South Australian Certificate of Education

1

Specialist Mathematics
2018

Question booklet 1

- **Part 1** (Questions 1 to 10) 75 marks
- Answer ***all*** questions in Part 1
- Write your answers in this question booklet
- You may write on page 24 if you need more space
- Allow approximately 90 minutes
- Approved calculators may be used — complete the box below

Examination information

Materials
- Question booklet 1 (Part 1)
- Question booklet 2 (Part 2)
- SACE registration number label

Reading time
- 10 minutes
- You may begin writing during this time
- You may begin using an approved calculator during this time

Writing time
- 3 hours
- Show appropriate working and steps of logic in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams

Total marks 150

Government of South Australia

Attach your SACE registration number label here

Graphics calculator

1. Brand ______________

Model ______________

2. Brand ______________

Model ______________

SACE
BOARD
OF SOUTH
AUSTRALIA

Remove this page from the booklet by tearing along the perforations.

MATHEMATICAL FORMULAE FOR USE IN STAGE 2 SPECIALIST MATHEMATICS

Circular functions

$\sin^2 A + \cos^2 A = 1$

$\tan^2 A + 1 = \sec^2 A$

$1 + \cot^2 A = \operatorname{cosec}^2 A$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$\quad = 2\cos^2 A - 1$

$\quad = 1 - 2\sin^2 A$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

$\sin A \pm \sin B = 2 \sin \frac{1}{2}(A \pm B) \cos \frac{1}{2}(A \mp B)$

$\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$

$\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$

Matrices and determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = |A| = ad - bc$ and

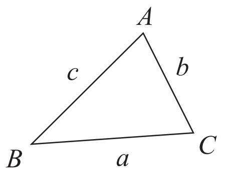
$A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Measurement

Area of sector, $A = \frac{1}{2} r^2 \theta$, where θ is in radians.

Arc length, $l = r\theta$, where θ is in radians.

In any triangle ABC:



Area of triangle, $A = \frac{1}{2} ab \sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$a^2 = b^2 + c^2 - 2bc \cos A$

Quadratic equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Distance from a point to a plane

The distance from (x_1, y_1, z_1) to $Ax + By + Cz + D = 0$ is given by

$\dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

Derivatives

$f(x) = y$	$f'(x) = \dfrac{dy}{dx}$
$\arcsin x$	$\dfrac{1}{\sqrt{1 - x^2}}$
$\arccos x$	$\dfrac{-1}{\sqrt{1 - x^2}}$
$\arctan x$	$\dfrac{1}{1 + x^2}$

Properties of derivatives

$\dfrac{d}{dx}\big(f(x)\,g(x)\big) = f'(x)g(x) + f(x)g'(x)$

$\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

$\dfrac{d}{dx} f(g(x)) = f'(g(x))\,g'(x)$

Arc length along a parametric curve

$l = \displaystyle\int_a^b \sqrt{\mathbf{v} \cdot \mathbf{v}}\, dt$, where $a \le t \le b$.

Integration by parts

$\displaystyle\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$

Volumes of revolution

About x axis, $V = \displaystyle\int_a^b \pi y^2\,dx$, where y is a function of x.

About y axis, $V = \displaystyle\int_c^d \pi x^2\,dy$, where y is a one-to-one function of x.

The examination questions begin on page 6.

PART 1 (Questions 1 to 10)

(75 marks)

Question 1 (5 marks)

Consider the following set of parametric equations:

$$\begin{cases} x(t) = t^3 - 3t + 1 \\ y(t) = e^{3t} \end{cases} \quad \text{where } 0 \le t \le \frac{3}{2}.$$

The graph of the curve defined by these parametric equations is shown in Figure 1 below.

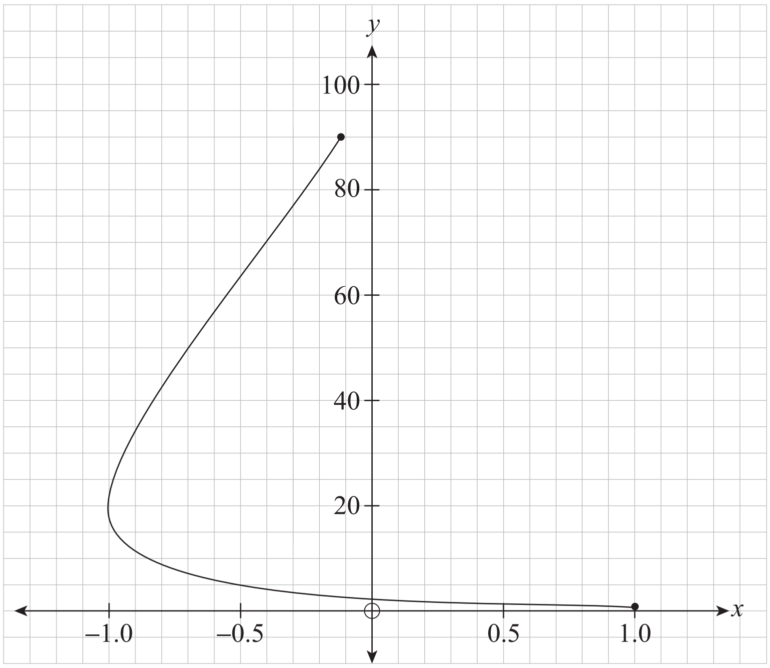


Figure 1

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{e^{3t}}{t^2 - 1}$.

(3 marks)

(b) Find the exact coordinates of the point (x, y) where $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined.

(2 marks)

Question 2 (6 marks)

(a) (i) Express $1+i$ in exact polar form.

(1 mark)

(ii) Express $\sqrt{3}-i$ in exact polar form.

(1 mark)

(b) Use your answers to part (a) to show that $(1+i)\left(\sqrt{3}-i\right)=2\sqrt{2}\operatorname{cis}\frac{\pi}{12}$.

(2 marks)

(c) Hence show that the exact value of $\cos\frac{\pi}{12}$ is $\frac{\sqrt{6}+\sqrt{2}}{4}$.

(2 marks)

Question 3 (6 marks)

Let $P(x)$ be a real cubic polynomial.

(a) If $2-i$ is a zero of $P(x)$, find a real quadratic factor of $P(x)$.

(2 marks)

(b) When $P(x)$ is divided by $(x-1)$ the remainder is 6, and when $P(x)$ is divided by $(x-2)$ the remainder is 5.

Find $P(x)$.

(4 marks)

Question 4 (9 marks)

Consider three planes in space that are defined by the following system of equations:

$$\begin{aligned} P_1 &: \quad x - 2y + z = 4 \\ P_2 &: \quad x + 3y - z = 0 \\ P_3 &: \quad 2x - y + az = a \end{aligned}$$

where a is a real constant.

(a) Write the system of equations in augmented matrix form.

(1 mark)

(b) Clearly stating all row operations, show that the augmented matrix reduces to

$$\begin{bmatrix} 1 & -2 & 1 & : & 4 \\ 0 & 5 & -2 & : & -4 \\ 0 & 0 & (4-5a) & : & (28-5a) \end{bmatrix}.$$

(3 marks)

(c) (i) Solve the system of equations for $a = 0$.

(2 marks)

(ii) Solve the system of equations for $a = \frac{4}{5}$.

(1 mark)

(iii) Which *one* of the following figures best represents the configuration of the three planes when $a = \frac{4}{5}$? Justify your answer.

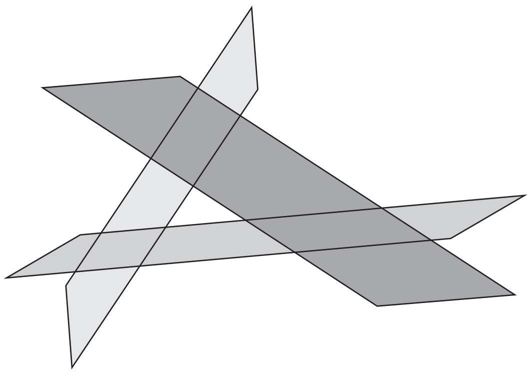


Figure 2

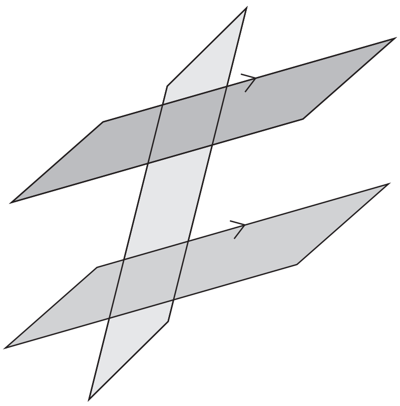


Figure 3

(2 marks)

Question 5 (9 marks)

(a) Figure 4 shows the points $A(1, 2, -3)$, $B(5, 3, -2)$, $C(6, 7, -3)$, and $D(2, 6, -4)$.

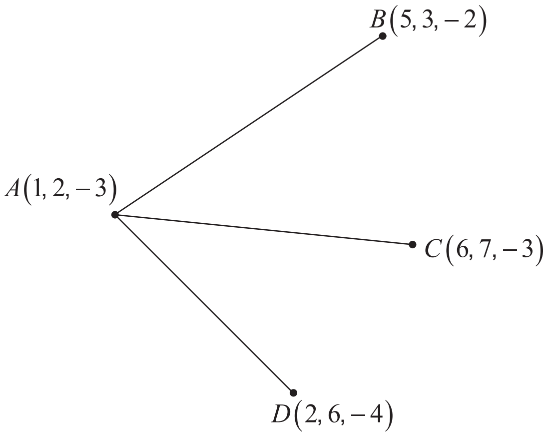


Figure 4

(i) Find $\overrightarrow{AB} \bullet \overrightarrow{AC}$.

(2 marks)

(ii) Find $\cos \angle BAC$.

(1 mark)

(iii) Find $\cos \angle CAD$.

(1 mark)

(b) Let $\overrightarrow{OP} = \boldsymbol{p}$ and $\overrightarrow{OQ} = \boldsymbol{q}$.

(i) On Figure 5, clearly show the vector $\overrightarrow{OR} = \boldsymbol{p} + \boldsymbol{q}$.

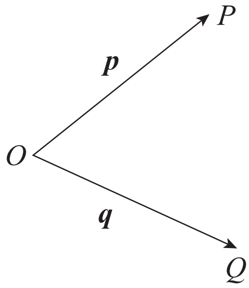


Figure 5

(1 mark)

(ii) If $|\boldsymbol{p}| = |\boldsymbol{q}|$, prove that $\overrightarrow{OR}$ bisects $\angle POQ$.

(2 marks)

(c) Figure 6 shows $\overrightarrow{OE} = [2, 5, -7]$ and $\overrightarrow{OF} = [10, 14, 4]$. Find a vector $\overrightarrow{OG}$ that bisects $\angle EOF$.

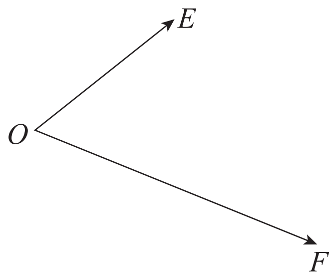


Figure 6

(2 marks)

 PLEASE TURN OVER

Question 6 (7 marks)

An object moving in a straight line has velocity given by the function

$$v(t) = e^{\cos^2 t} \sin 2t$$

where v is the velocity of the object measured in metres per second and t is time measured in seconds.

(a) On the axes in Figure 7, sketch the graph of $v(t)$ for $0 \le t \le \pi$.

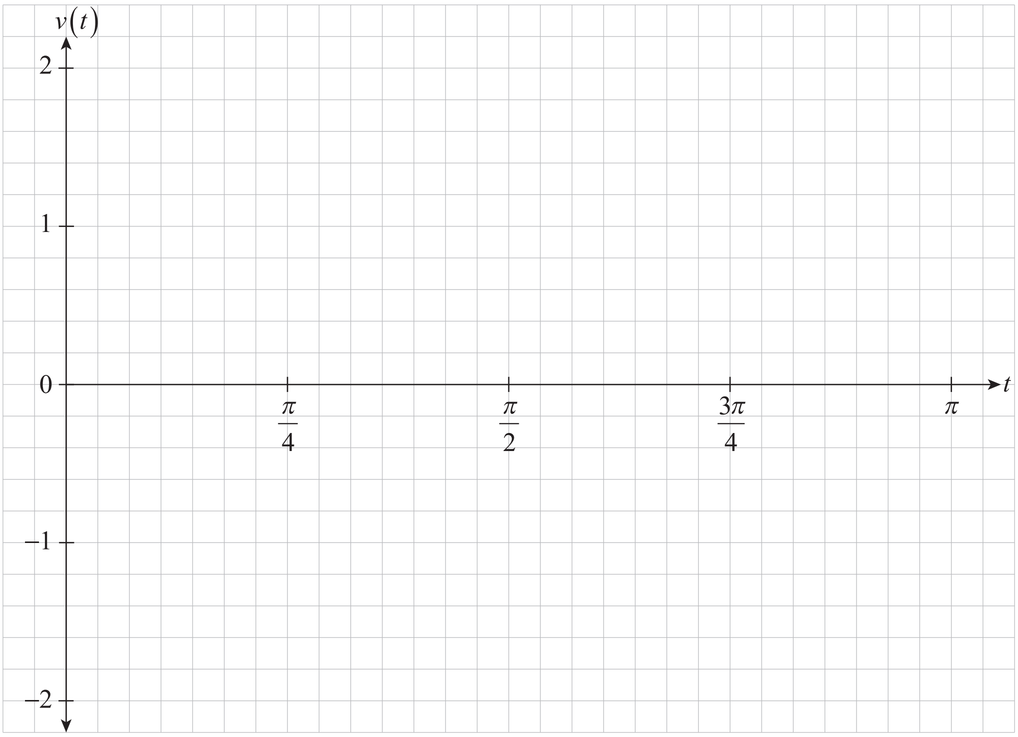


Figure 7 (3 marks)

(b) Find the exact distance travelled by the object between $t = 0$ and $t = \frac{\pi}{2}$ seconds.

(3 marks)

(c) Hence find the exact total distance travelled by the object between $t = 0$ and $t = \pi$ seconds.

(1 mark)

Question 7 (7 marks)

(a) Verify that $\frac{x^2}{x+1} = (x-1) + \frac{1}{x+1}$.

(1 mark)

(b) Use integration by parts to show that, for $x > -1$:

$$\int x \ln(x+1)\,\mathrm{d}x = \frac{x^2}{2}\ln(x+1) - \frac{1}{4}(x-1)^2 - \frac{1}{2}\ln(x+1) + c$$

where c is a constant.

(3 marks)

(c) (i) Figure 8 shows the graph of $f(x) = x\ln(x+1)$ for $x > -1$.

On the same axes, sketch the graph of $g(x) = x|\ln(x+1)|$ for $x > -1$.

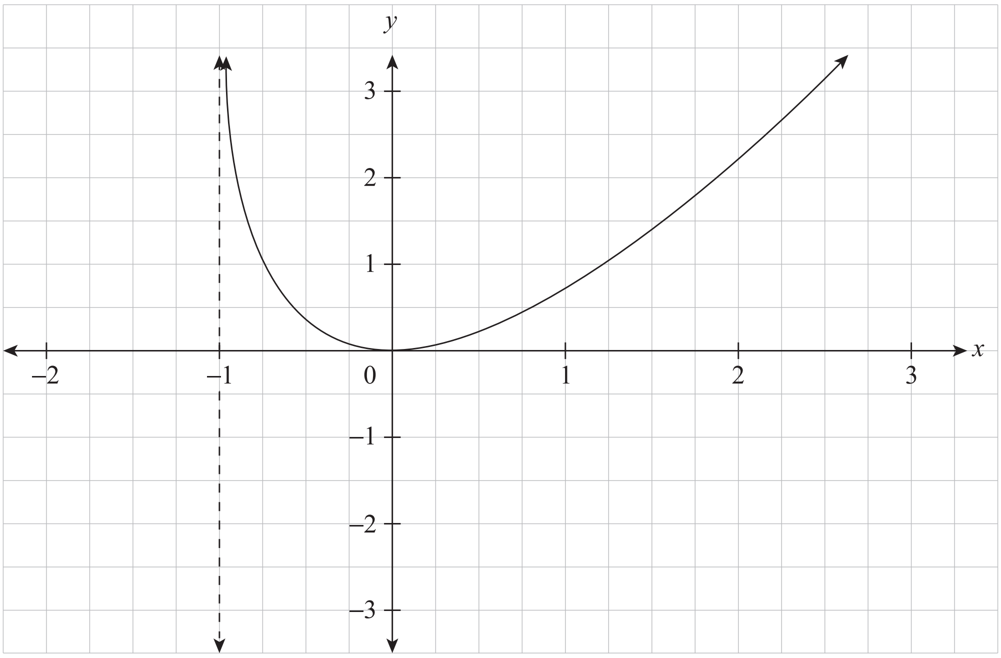


Figure 8

(1 mark)

(ii) Using the information in part (b), find the exact value of $\int_{-\frac{1}{2}}^{0} g(x)\,\mathrm{d}x$.

(2 marks)

Question 8 (7 marks)

(a) Prove by mathematical induction that $10^n - 6^n$ is divisible by 4 for all positive integers n.

(5 marks)

(b) Hence find the last two digits of $25\left(10^{2018}-6^{2018}\right)$.

(2 marks)

Question 9 (9 marks)

The rate at which the concentration, C, of a chemical changes during a series of reactions is given by

$$\frac{\mathrm{d}C}{\mathrm{d}t} = b - kC$$

where b and k are positive real constants, and t is time measured in microseconds.

(a) If initially C is zero, and it is known that $b = 2.27$ and $k = 0.303$, use integration to show that

$$C \approx -7.49\left(e^{-0.303t} - 1\right).$$

(4 marks)

(b) Find the approximate limiting concentration of the chemical as $t \to \infty$.

(1 mark)

(c) The concentration, A, of a different chemical is also dependent on the constant b, and can be modelled by the equation

$$A \approx 7.49\left(e^{-0.303t} + 1\right)$$

where t is time measured in microseconds.

(i) On the axes in Figure 9, draw the graph of A and the graph of C over time. Label both graphs.

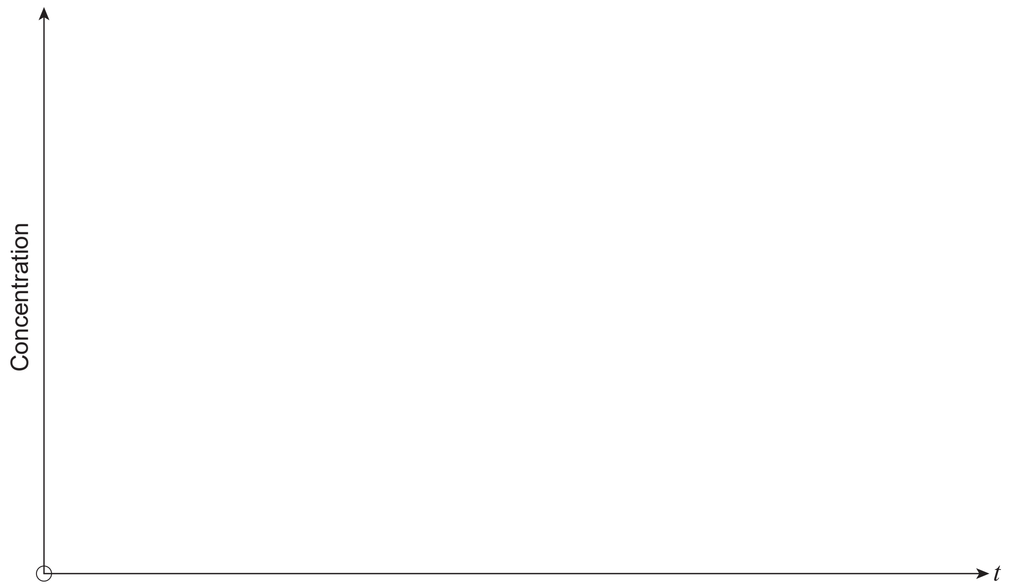


Figure 9

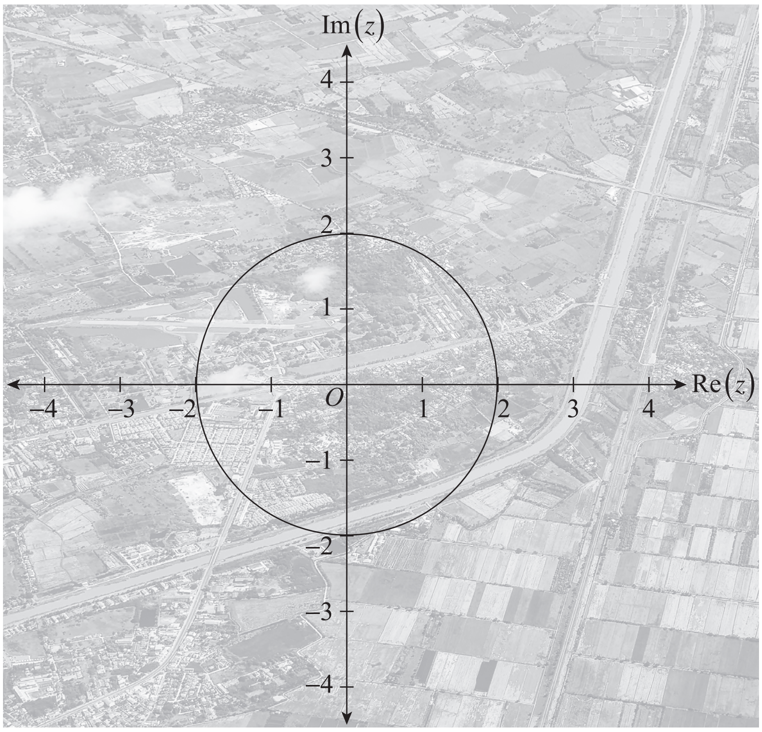
(3 marks)

(ii) On the graphs that you drew on the axes in Figure 9, indicate the approximate limiting concentration of each chemical as $t \to \infty$.

(1 mark)

Question 10 (10 marks)

Figure 10 shows an Argand diagram superimposed on an aerial photograph.



Source: © Suvit Maka | Dreamstime.com

Figure 10

(a) Write an inequality that represents all complex numbers z in the region bounded by, and including, the circle.

(2 marks)

(b) (i) On Figure 10, mark the position of the complex number $3+4i$ with an X. (1 mark)

(ii) Show that any point on the straight line through the origin (O) and X has the form $3t+4it$, where t is a real parameter.

(1 mark)

(c) (i) If z is any point in the region bounded by, and including, the circle, apply the triangle inequality to the triangle that has vertices at z, X, and O to show that

$$|z-(3+4i)| \leq 7.$$

(2 marks)

(ii) On Figure 10, on the region bounded by, and including, the circle, mark the point P for which $|z-(3+4i)|=7$. (1 mark)

(d) Using part (b)(ii) or otherwise, find the complex number that is represented by P.

(2 marks)

(e) A mobile phone tower at O provides reception for 2 km in any direction. A new tower is going to be built at X, which will provide reception for 7 km in any direction.

Explain why the tower at O will not be needed, once a tower is built at X.

(1 mark)

You may write on this page if you need more space to finish your answers to any of the questions in Part 1. Make sure to label each answer carefully (e.g. 7(c)(ii) continued).

South Australian Certificate of Education

Specialist Mathematics
2018

Question booklet 2

- **Part 2** (Questions 11 to 15) 75 marks
- Answer ***all*** questions in Part 2
- Write your answers in this question booklet
- You may write on page 19 if you need more space
- Allow approximately 90 minutes
- Approved calculators may be used — complete the box below

2

Government of South Australia

Copy the information from your SACE label here

SEQ	FIGURES						CHECK LETTER	BIN

Graphics calculator

1. Brand ____________

 Model ____________

2. Brand ____________

 Model ____________

PART 2 (Questions 11 to 15)

(75 marks)

Question 11 (15 marks)

The points $A(3,1,-1)$, $B(0,2,10)$, and $C(0,0,6)$ are on the plane P.

(a) (i) Find $\overrightarrow{AB}$.

(1 mark)

(ii) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(1 mark)

(iii) Show that the equation of P is $3x - 2y + z = 6$.

(2 marks)

(b) (i) Find the equation of the normal to P through the point $D(6,-1,0)$.

(2 marks)

(ii) Show that the normal found in part (b)(i) intersects P at $A(3,1,-1)$.

(2 marks)

(c) Figure 11 shows the point $E(3,0,11)$ on the normal to P through $B(0,2,10)$.

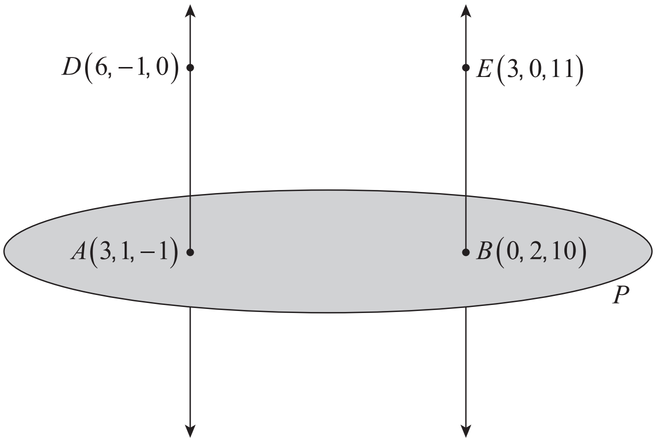


Figure 11

(i) Show that the line through D and E is parallel to P.

(2 marks)

(ii) Find the distance from this line to P.

(1 mark)

(d) (i) Show that the point $F(-3, 2, 5)$ is the same distance from P as the line that passes through D and E is from P.

(2 marks)

(ii) Is F on the line through D and E? Explain your answer.

(2 marks)

Question 12 (15 marks)

Figure 12 shows an amusement park ride viewed from above. The ride consists of a circular cup that rolls on the inside of the boundary of a larger circle.

The larger circle has centre O and radius 5 metres.

The following parametric equations describe the position of a person at P sitting in a cup of radius b metres:

$$\begin{cases} x = (5-b)\cos\theta + b\cos\left(\dfrac{5-b}{b}\theta\right) \\ y = (5-b)\sin\theta - b\sin\left(\dfrac{5-b}{b}\theta\right) \end{cases}$$

where $-2\pi < \theta \leq 2\pi$.

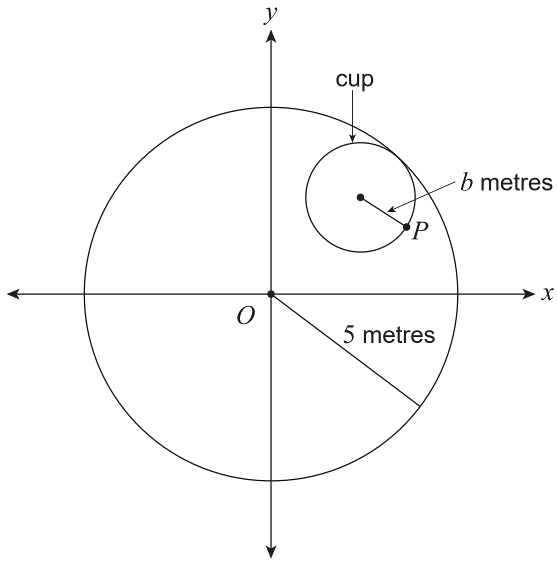


Figure 12

(a) Consider the case where $b = 1$.

(i) Find the parametric equations describing the position of a person at P sitting in a cup of radius 1 metre.

(1 mark)

(ii) On the axes in Figure 13, sketch the curve defined by the parametric equations that you found in part (a)(i).

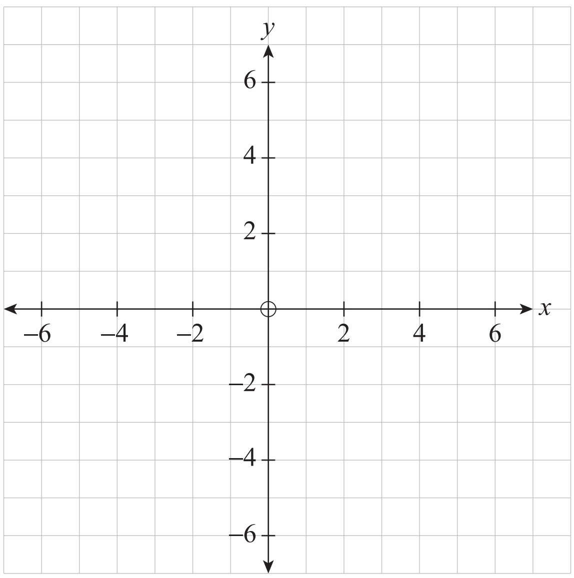


Figure 13

(3 marks)

(b) Now consider the case where $b = 2$.

Figure 14 shows the graph of the curve defined by the parametric equations that describe the position of a person at P for $-2\pi < \theta \le 2\pi$.

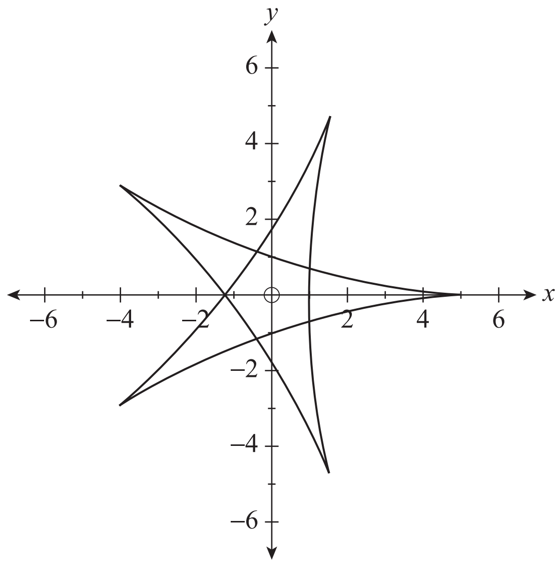


Figure 14

(i) Show that the parametric equations that describe the position of a person at P are

$$\begin{cases} x = 3\cos\theta + 2\cos(1.5\theta) \\ y = 3\sin\theta - 2\sin(1.5\theta) \end{cases} \quad \text{where } -2\pi < \theta \le 2\pi.$$

(1 mark)

(ii) Find the velocity vector, $\boldsymbol{v}$, of the person at P when $\theta = \pi$.

(3 marks)

For particular values of θ, corresponding to the vertices of the graph drawn in Figure 14, the velocity vector is not defined. You may assume that the values of θ considered in the following questions do not correspond to these vertices.

(iii) Show that the speed of the person at any point that is ***not*** a vertex, is given by

$$S = \sqrt{18 - 18\cos(2.5\theta)}.$$

(3 marks)

(iv) Find the coordinates of P when the cup reaches maximum speed during the interval

$$0 < \theta < \frac{4\pi}{5}.$$

(2 marks)

(v) (1) Write an expression for the distance the person has travelled between $\theta = \frac{\pi}{5}$ and $\theta = \frac{3\pi}{5}$.

(1 mark)

(2) *Hence* find this distance.

(1 mark)

Question 13 (14 marks)

A model was developed to describe the probability that a spider of a particular species dies from a dose of a toxic substance. The probability, Y, that a spider dies is given by

$$Y = \frac{e^{-1.65+1.12x}}{1+e^{-1.65+1.12x}}$$

where x is the dose of the toxic substance in arbitrary units.

(a) Integrate the following differential equation to find the formula for Y:

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \frac{1.12e^{-1.65+1.12x}}{\left(1+e^{-1.65+1.12x}\right)^2}\text{, where } Y(0) = \frac{e^{-1.65}}{1+e^{-1.65}}.$$

(6 marks)

(b) On the slope field in Figure 15, draw the solution curve for the initial condition $Y(0) = \dfrac{e^{-1.65}}{1+e^{-1.65}}$.

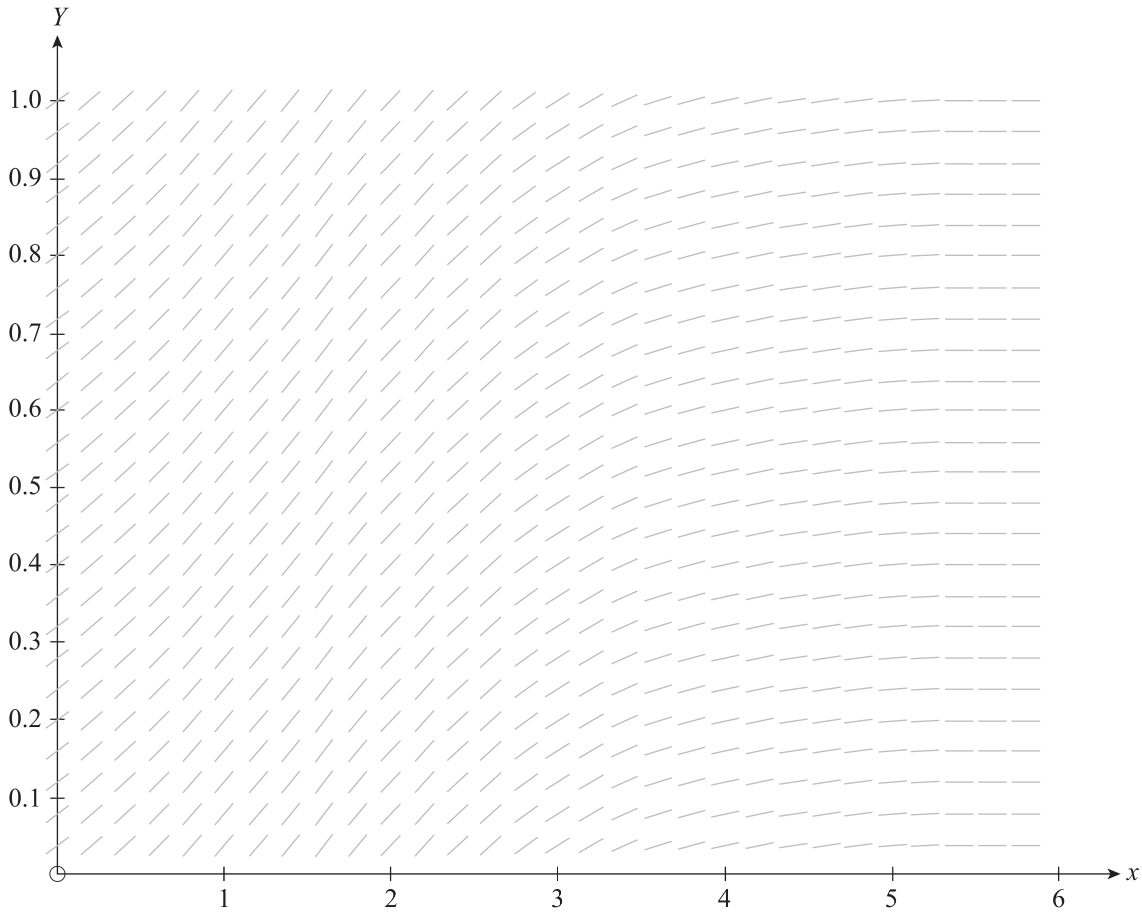


Figure 15

(3 marks)

(c) Find the dose required if the probability that a spider dies is 0.5. Give your answer correct to three significant figures.

(2 marks)

(d) Find the dose required if

(i) $Y = 0.99$. Give your answer correct to three significant figures.

(1 mark)

(ii) $Y = 0.999$. Give your answer correct to three significant figures.

(1 mark)

(e) Consider $Y = \dfrac{e^{-1.65 + ax}}{1 + e^{-1.65 + ax}}$, where a is a positive constant related to the strength of the toxic substance.

For the value of x that you found in part (d)(i), find the value of a that increases the probability that a spider dies from 0.99 to 0.999. Give your answer correct to three significant figures.

(1 mark)

Question 14 (16 marks)

(a) (i) Show that $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$.

(1 mark)

(ii) Figure 16 shows the graph of $g(x) = \sin x + \cos x$ for $-\pi \le x \le \pi$.

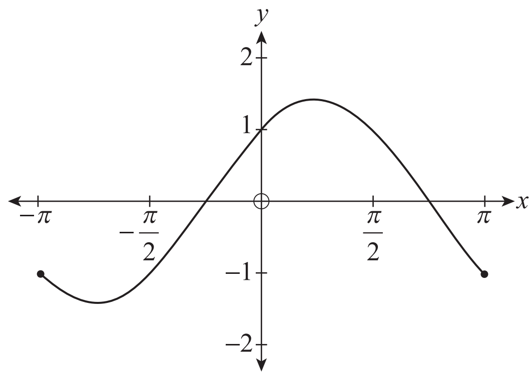


Figure 16

Explain why $g(x)$ is a function, but does not have an inverse function.

(2 marks)

(iii) Explain why the following function *does* have an inverse function:

$$f(x) = \sin x + \cos x \text{ where } -\frac{3\pi}{4} \le x \le \frac{\pi}{4}.$$

(1 mark)

(iv) Show that $f^{-1}(x) = \arcsin\left(\frac{x}{\sqrt{2}}\right) - \frac{\pi}{4}$.

(2 marks)

(b) Figure 17 shows the graph of the inverse function $f^{-1}(x)$.

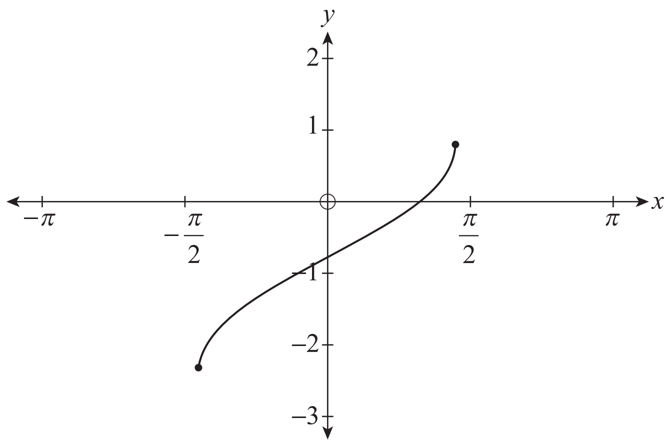


Figure 17

State the domain and range of $f^{-1}(x)$ in exact form.

(2 marks)

(c) If $y = \arcsin\left(\frac{x}{\sqrt{2}}\right)$, then $\sin y = \frac{x}{\sqrt{2}}$.

Using implicit differentiation, show that

$$\frac{dy}{dx} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{1 - \frac{x^2}{2}}}.$$

(3 marks)

Consider a wall brace leaning against a building. The bottom of the wall brace is 5 metres along the ground from the base of the building, and the top of the wall brace is 5 metres above the ground, as shown in Figure 18.

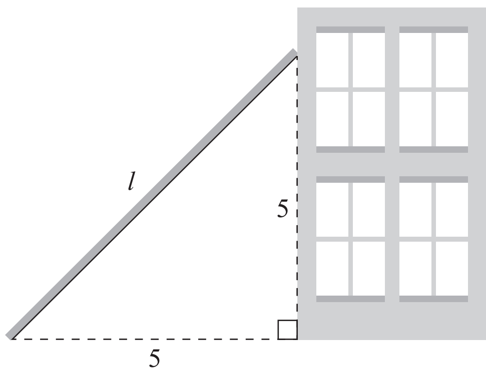


Figure 18

(d) Show that the length of the wall brace, l, is $5\sqrt{2}$ metres.

(1 mark)

The top of this wall brace slides down the side of the building. The top of the wall brace is now x metres above the ground, and θ is the angle of inclination of the wall brace with respect to the ground, as shown in Figure 19.

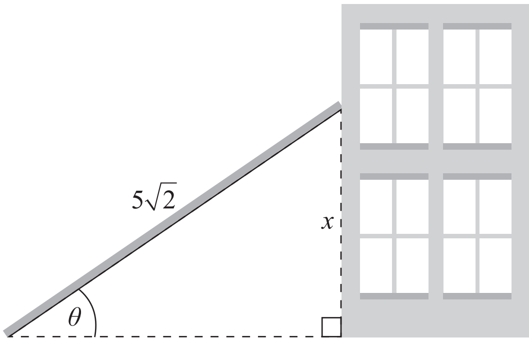


Figure 19

(e) (i) Show that $\dfrac{d\theta}{dt} = \dfrac{\frac{1}{5\sqrt{2}}\frac{dx}{dt}}{\sqrt{1-\frac{x^2}{50}}}$.

(2 marks)

(ii) If the top of the wall brace slides down the side of the building at a rate of 0.05 metres per second, at what rate is θ changing when the bottom of the wall brace is 6 metres along the ground from the base of the building?

(2 marks)

Question 15 (15 marks)

(a) Let $z = x + iy$ be a complex number.

(i) Find all solutions of the equation $\overline{z} = z^{-1}$, where $\overline{z}$ is the conjugate of z.

(2 marks)

(ii) Plot the solutions on the Argand diagram in Figure 20.

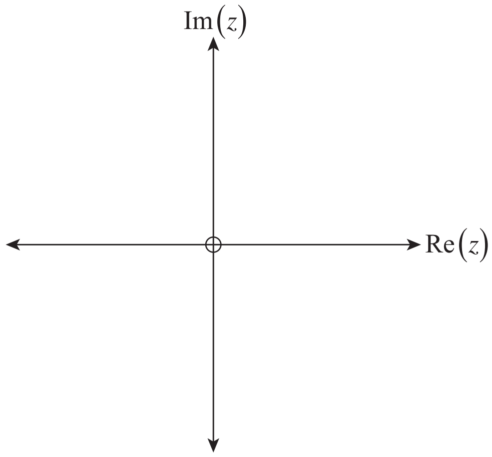


Figure 20 (1 mark)

(b) Now let $z = \text{cis}\,\theta$ be a complex number with modulus 1.

(i) Using de Moivre's theorem, show that $z^n + z^{-n} = 2\cos n\theta$.

(2 marks)

(ii) Expand $\left(z+z^{-1}\right)^3$ to show that $\cos^3\theta = \frac{1}{4}\left(\cos 3\theta + 3\cos\theta\right)$.

(3 marks)

(iii) Hence find the exact value of $\int_0^{\frac{\pi}{4}} \cos^3\theta \, d\theta$.

(3 marks)

(c) Let $f(x) = x^{\frac{3}{2}}$ and $g(\theta) = \cos\theta$ be functions of a real variable, and let $h(\theta) = f\left(g(\theta)\right)$.

(i) Find the formula for $h(\theta)$.

(1 mark)

(ii) On the axes in Figure 21, sketch the graph of $h(\theta)$ on the largest interval containing $\theta = 0$ for which the function is defined.

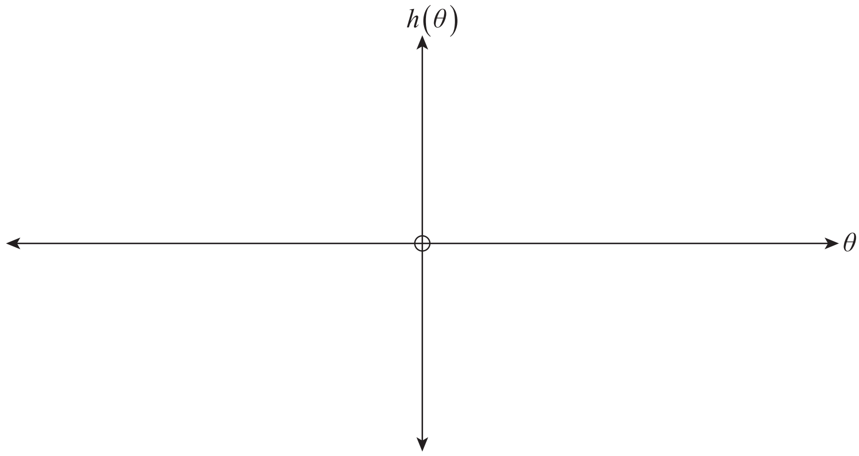


Figure 21

(2 marks)

(iii) Using your answer to part (b)(iii), find the exact volume of the solid that is obtained when the region of the graph of $h(\theta)$ that is bounded by the lines $\theta = 0$ and $\theta = \frac{\pi}{4}$ is rotated about the θ-axis.

(1 mark)

You may write on this page if you need more space to finish your answers to any of the questions in Part 2. Make sure to label each answer carefully (e.g. 12(b)(iii) continued).