1

Government of South Australia

SACE Board of SA | South Australian Certificate of Education

Specialist Mathematics
2017

Question Booklet 1

- **Part 1** (Questions 1 to 10) 75 marks
- Answer ***all*** questions in Part 1
- Write your answers in this question booklet
- You may write on page 26 if you need more space
- Allow approximately 90 minutes
- Approved calculators may be used — complete the box below

Examination information

Materials
- Question Booklet 1 (Part 1)
- Question Booklet 2 (Part 2)
- SACE registration number label

Reading time
- 10 minutes
- You may make notes on scribbling paper

Writing time
- 3 hours
- Show all working in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams

Total marks 150

Attach your SACE registration number label here

Graphics calculator

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You may remove this page from the booklet by tearing along the perforations.

LIST OF MATHEMATICAL FORMULAE FOR USE IN STAGE 2 SPECIALIST MATHEMATICS

Circular Functions

$\sin^2 A + \cos^2 A = 1$

$\tan^2 A + 1 = \sec^2 A$

$1 + \cot^2 A = \operatorname{cosec}^2 A$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned}$$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

$\sin A \pm \sin B = 2 \sin \frac{1}{2}(A \pm B) \cos \frac{1}{2}(A \mp B)$

$\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$

$\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$

Matrices and Determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = |A| = ad - bc$ and

$A^{-1} = \dfrac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Quadratic Equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Distance from a Point to a Plane

The distance from (x_1, y_1, z_1) to $Ax + By + Cz + D = 0$ is given by

$\dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

Derivatives

$f(x) = y$	$f'(x) = \dfrac{dy}{dx}$
$\arcsin x$	$\dfrac{1}{\sqrt{1 - x^2}}$
$\arccos x$	$\dfrac{-1}{\sqrt{1 - x^2}}$
$\arctan x$	$\dfrac{1}{1 + x^2}$

Properties of Derivatives

$\dfrac{d}{dx}\big(f(x)\,g(x)\big) = f'(x)g(x) + f(x)g'(x)$

$\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

$\dfrac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$

Integration by Parts

$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$

Volumes of Revolution

About x axis $\int_a^b \pi y^2\,dx$, where y is a function of x.

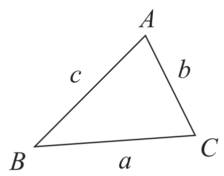
About y axis $\int_c^d \pi x^2\,dy$, where y is a one-to-one function of x.

Mensuration

Area of sector $= \frac{1}{2} r^2 \theta$, where θ is in radians.

Arc length $= r\theta$, where θ is in radians.

In any triangle ABC:



Area of triangle $= \frac{1}{2} ab \sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$a^2 = b^2 + c^2 - 2bc \cos A$

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The examination questions begin on page 6.

PART 1 (Questions 1 to 10)

(75 marks)

QUESTION 1 (5 marks)

(a) Find $\int \cos t \sin^2 t \, \mathrm{d}t$.

(2 marks)

(b) A building in a small town casts a shadow during daylight hours. The area of the shadow changes depending on the time of day. The rate of change of the area of the shadow can be modelled by the differential equation

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -2t + 150 \cos t \sin^2 t$$

where A is the area of the shadow in square metres and t is the time in hours. At sunrise, $t = 0$ and $A = 375 \text{ m}^2$.

Solve the differential equation, and hence find an expression for the area of the shadow at time t.

(3 marks)

QUESTION 2 (6 marks)

Polynomial $S(x)$ has a remainder of $x+1$ when divided by x^2+x-2.

(a) (i) Write $S(x)$ in the form $S(x)=Q(x)D(x)+R(x)$, where $Q(x)$ is the quotient, $D(x)$ is the divisor, and $R(x)$ is the remainder.

(2 marks)

(ii) Find the remainder when $S(x)$ is divided by $x+2$.

(2 marks)

(b) If $P(x)=S(x)-T(x)$, where $T(x)$ is a polynomial and $T(-2)=-1$, show that $x=-2$ is a zero of the polynomial $P(x)$.

(2 marks)

QUESTION 3 (6 marks)

Figure 1 shows rhombus $OPQR$ with $\overrightarrow{OP} = \boldsymbol{p}$ and $\overrightarrow{OR} = \boldsymbol{r}$.

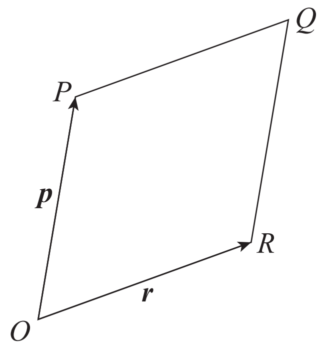


Figure 1

(a) (i) Find $\overrightarrow{OQ}$ in terms of $\boldsymbol{p}$ and $\boldsymbol{r}$.

(1 mark)

(ii) Find $\overrightarrow{PR}$ in terms of $\boldsymbol{p}$ and $\boldsymbol{r}$.

(1 mark)

(b) (i) Show that $\overrightarrow{OQ} \bullet \overrightarrow{PR} = |\boldsymbol{r}|^2 - |\boldsymbol{p}|^2$.

(2 marks)

(ii) Hence prove that the diagonals of the rhombus $OPQR$ are perpendicular, giving reasons.

(2 marks)

QUESTION 4 (7 marks)

A curve has the following parametric equations:

$$\begin{cases} x(t) = \sqrt{\cos t} \\ y(t) = \sin t \end{cases} \quad \text{where } 0 \leq t \leq \frac{\pi}{2}.$$

(a) Draw a graph of this curve on the axes in Figure 2.

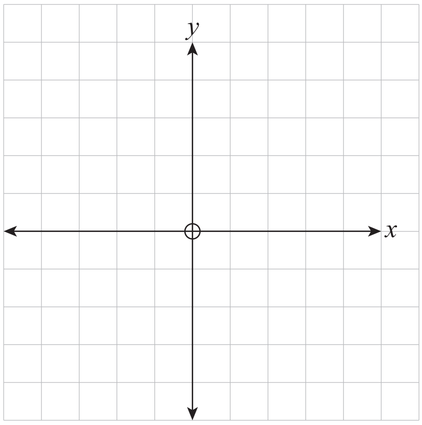


Figure 2

(3 marks)

(b) Show that all points (x, y) on the curve that you drew in Figure 2 satisfy the equation $x^4 + y^2 = 1$.

(1 mark)

(c) Use implicit differentiation to show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x^3}{y}$, where $y \neq 0$.

(2 marks)

(d) Find the slope of the tangent to the curve at $t = \frac{\pi}{6}$.

(1 mark)

QUESTION 5 (8 marks)

(a) Use mathematical induction to prove that:

$$\frac{1}{4\times1^2-1}+\frac{1}{4\times2^2-1}+\ldots+\frac{1}{4\times n^2-1}=\frac{n}{2n+1}$$, where n is a positive integer.

(6 marks)

(b) Hence find, in simplest rational form, the value of $\frac{1}{15}+\frac{1}{35}+\ldots+\frac{1}{399}$.

(2 marks)

QUESTION 6 (7 marks)

Figure 3 shows the area of grass, A, cut by a single sweep of a tool called a 'scythe', when used by a person standing at position O.

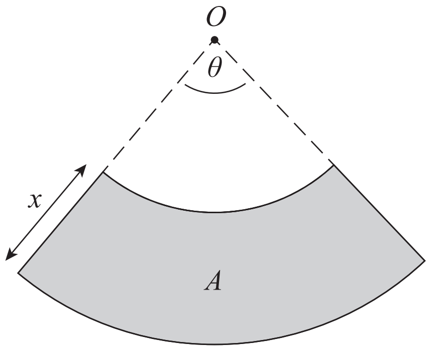
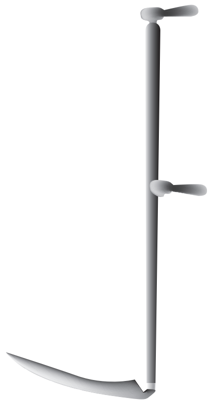


Figure 3

This photograph cannot be reproduced here for copyright reasons.



Scythe

The area of cut grass is $A=\frac{1}{2}\left(x^2+2x\right)\theta$, where x is the width (in metres) of the area of cut grass and θ is the angle (in radians) through which the scythe moves in a single sweep.

(a) Show that $\frac{\mathrm{d}A}{\mathrm{d}t}=(x+1)\theta\frac{\mathrm{d}x}{\mathrm{d}t}+\left(\frac{x^2}{2}+x\right)\frac{\mathrm{d}\theta}{\mathrm{d}t}$.

(3 marks)

(b) Consider the instant when $x = 0.5$ m and $A = 1$ m^2.

(i) Find the value of θ.

(1 mark)

(ii) Using the information in part (a) and your answer to part (b)(i), find $\frac{dA}{dt}$ at the instant when $\frac{d\theta}{dt} = \frac{2\pi}{3}$ radians per second and when x is increasing at 0.2 m s^{-1}.

Give your answer correct to four significant figures.

(3 marks)

QUESTION 7 (8 marks)

Figure 4 shows a circle with centre A in the complex plane.

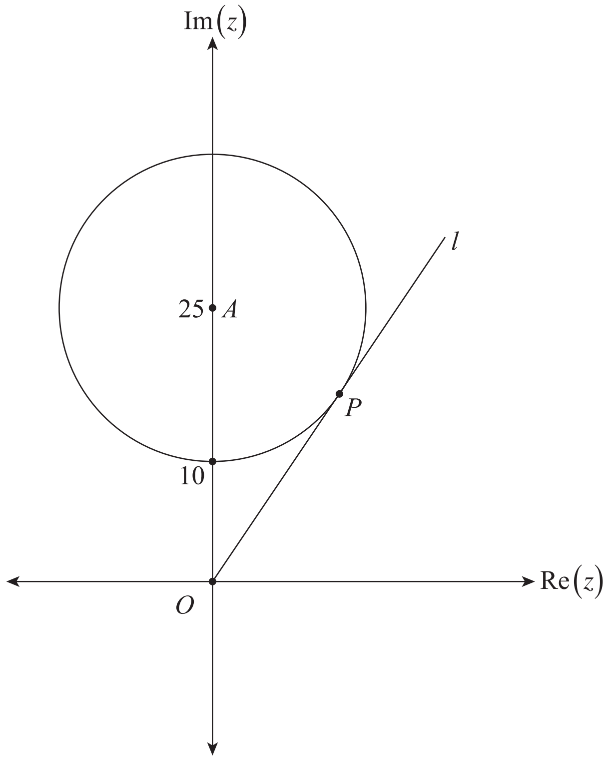


Figure 4

(a) Write down an equation, in terms of z, that describes exactly all points on the circumference of the circle in Figure 4.

(2 marks)

(b) The line l through the origin O is tangent to the circle at the point P.

The point P represents the complex number w.

(i) Show that $|w| = 20$.

(2 marks)

(ii) Show that $\arg w = \angle OAP$.

(2 marks)

(iii) Hence write w in the form $a + b\mathrm{i}$.

(2 marks)

QUESTION 8 (10 marks)

(a) Show that $\dfrac{1}{x-2}-\dfrac{1}{x+3}=\dfrac{5}{(x-2)(x+3)}$.

(1 mark)

Let $f(x)=\dfrac{5}{(x-2)(x+3)}$.

(b) (i) Draw the graph of $y=f(x)$ on the axes in Figure 5.

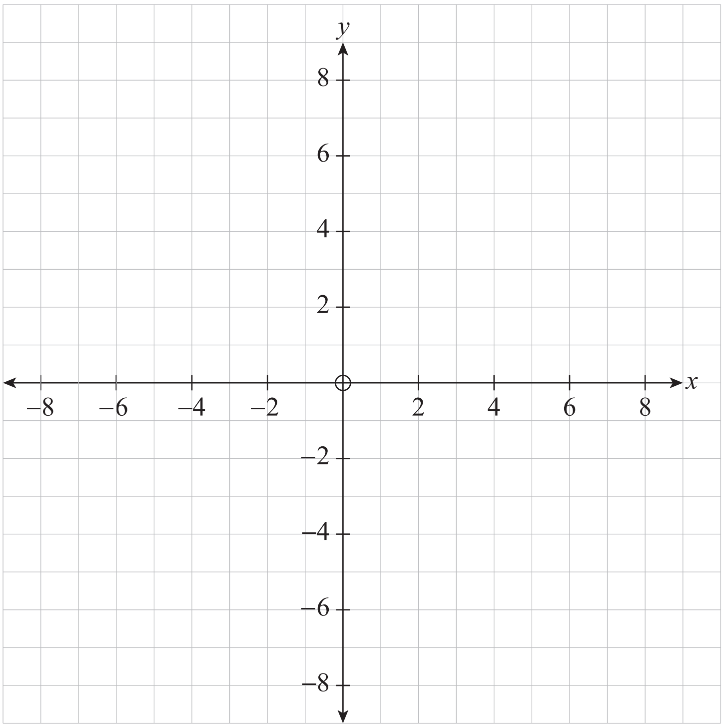


Figure 5

(3 marks)

(ii) Draw the graph of $y = |f(x)|$ on the axes in Figure 6.

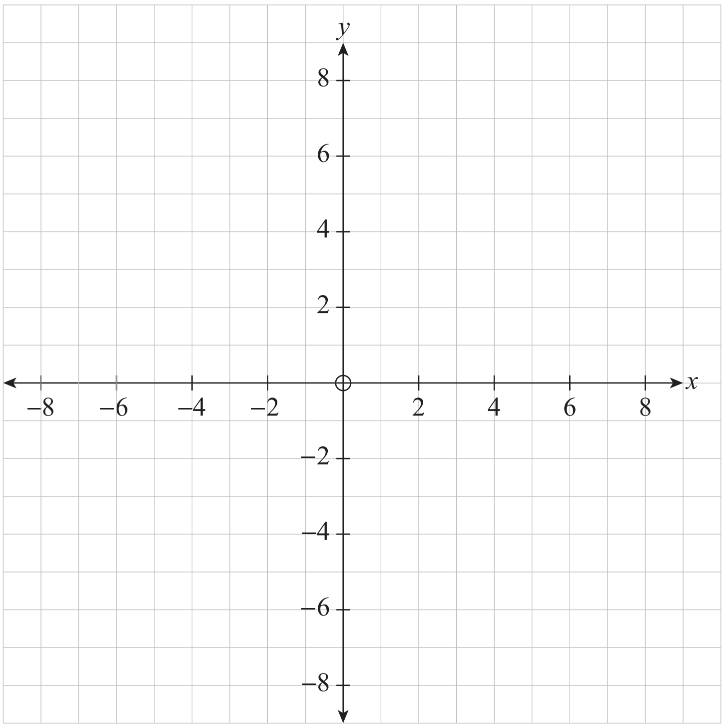


Figure 6

(1 mark)

(iii) Draw the graph of $y = |f(x)| - f(x)$ on the axes in Figure 7.

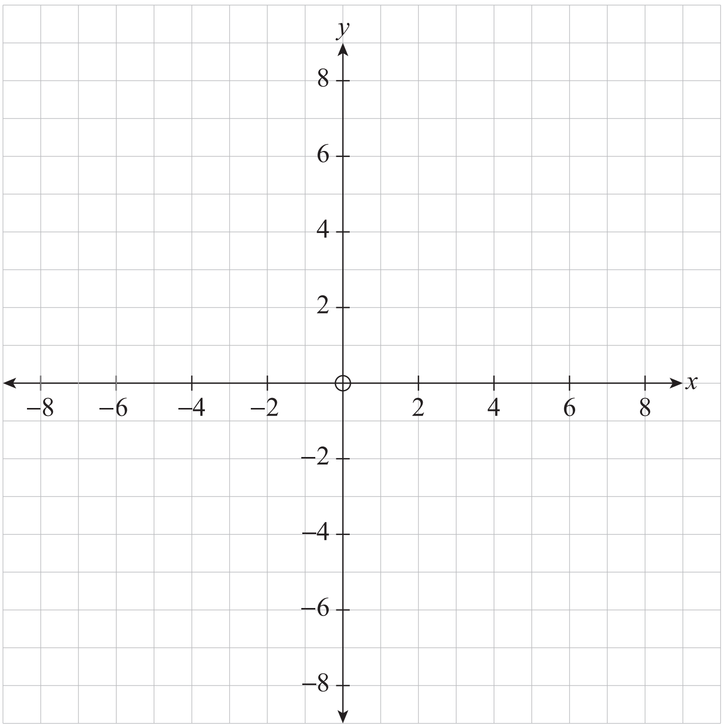


Figure 7

(2 marks)

(c) Find the exact area between the graph of $y = \left| f\left(x\right) \right| - f\left(x\right)$, the x-axis, and the lines $x = -2$ and $x = 1$.

(3 marks)

Question 9 begins on page 22.

QUESTION 9 (8 marks)

(a) (i) Use integration by parts to find $\int xe^{2x}\mathrm{d}x$.

(3 marks)

(ii) Use integration by parts to show that

$$\int x^2e^{2x}\mathrm{d}x = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c$$, where c is a constant.

(2 marks)

(b) Let $f(x) = xe^x$.

The graph of $y = f(x)$ for $x \geq 0$ is shown in Figure 8.

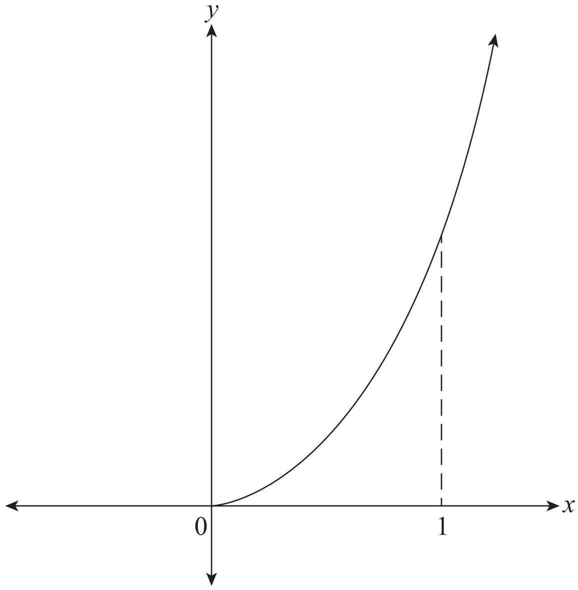


Figure 8

Find the exact volume of the solid obtained when the region bounded by the graph of $f(x)$ on the interval $[0, 1]$ is rotated about the x-axis.

(3 marks)

QUESTION 10 (10 marks)

Figure 9 shows points $O(0,0,0)$, $P(1, \sin\theta, \cos\theta)$, and $Q(\sqrt{2},1,1)$. The vector $\boldsymbol{a} = \overrightarrow{OP}$ and the vector $\boldsymbol{b} = \overrightarrow{PQ}$.

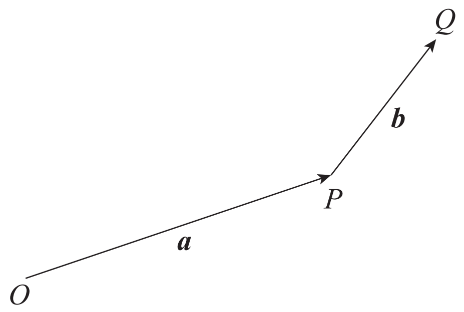


Figure 9

(a) (i) On Figure 9, draw and label the vector $\boldsymbol{a}+\boldsymbol{b}$. (1 mark)

(ii) Calculate $|\boldsymbol{a}+\boldsymbol{b}|$.

(1 mark)

(b) (i) Show that $|\boldsymbol{a}|+|\boldsymbol{b}| = \sqrt{2}+\sqrt{6-2\sqrt{2}-2(\sin\theta+\cos\theta)}$.

(3 marks)

(ii) State why $2 \leq \sqrt{2} + \sqrt{6 - 2\sqrt{2} - 2(\sin\theta + \cos\theta)}$.

(1 mark)

(c) (i) State the relationship between $\boldsymbol{a}$ and $\boldsymbol{b}$ when $2 = \sqrt{2} + \sqrt{6 - 2\sqrt{2} - 2(\sin\theta + \cos\theta)}$.

(1 mark)

(ii) Hence find an exact value of θ for which $2 = \sqrt{2} + \sqrt{6 - 2\sqrt{2} - 2(\sin\theta + \cos\theta)}$.

(2 marks)

(iii) Hence show that $2 = \sqrt{2} + \sqrt{6 - 4\sqrt{2}}$.

(1 mark)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. Question 10(b)(i) continued).

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Question Booklet 2

2

- **Part 2** (Questions 11 to 15) 75 marks
- Answer ***all*** questions in Part 2
- Write your answers in this question booklet
- You may write on page 18 if you need more space
- Allow approximately 90 minutes
- Approved calculators may be used — complete the box below

Copy the information from your SACE label here

SEQ	FIGURES	CHECK LETTER	BIN

Graphics calculator

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Model ____________
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Model ____________

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PART 2 (Questions 11 to 15)

(75 marks)

QUESTION 11 (14 marks)

(a) A particle moves according to the following parametric equations:

$$\begin{cases} x(t) = t^2 \\ y(t) = t^3 - t \end{cases} \quad \text{where } -2 \le t \le 2.$$

(i) On the axes in Figure 10, draw a graph of the path travelled by the particle.

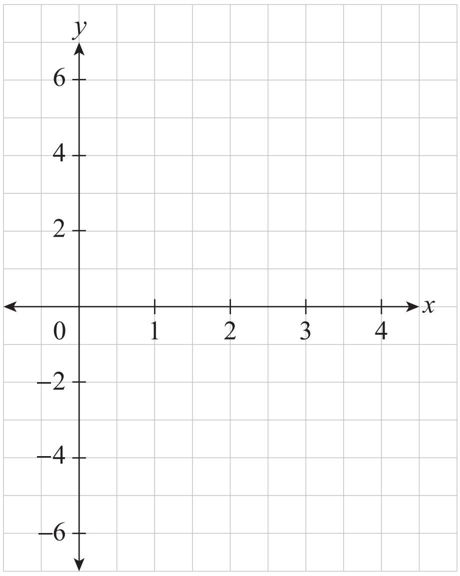


Figure 10

(3 marks)

(ii) Find the values of t when $y(t) = 0$.

(1 mark)

(iii) Find the length of the path travelled by the particle on the interval $-1 \le t \le 1$.

(3 marks)

(b) Consider a different particle moving according to the following parametric equations:

$$\begin{cases} x(t) = t^2 \\ y(t) = t^3 - 2t \end{cases} \quad \text{where } -2 \le t \le 2.$$

The graph of the path travelled by this particle is shown in Figure 11.

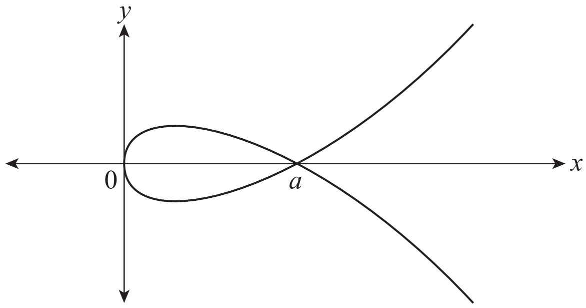


Figure 11

(i) Find the total length of the path between the x-intercepts $x = 0$ and $x = a$; that is, find the total length of the loop.

(3 marks)

(ii) (1) Show that the path travelled by this particle is given by $y = \pm\left(x^{\frac{3}{2}} - 2x^{\frac{1}{2}}\right)$.

(2 marks)

(2) Calculate $\displaystyle\int_0^2 \left(x^{\frac{3}{2}} - 2x^{\frac{1}{2}}\right) \mathrm{d}x$.

(1 mark)

(3) Hence find the area enclosed by the loop.

(1 mark)

QUESTION 12 (16 marks)

Consider the system of equations shown below.

$$\begin{cases} x+2y+2z=4 \\ 2x+y-2z=5 \\ 3x+2y-2z=8 \end{cases}$$

(a) (i) Write this system as an augmented matrix.

(1 mark)

(ii) Stating all row operations, show that this system of equations has the following solutions:

$$x=2+2t$$
$$y=1-2t$$
$$z=t$$

where t is a real parameter.

(3 marks)

(b) Consider three planes in space, P_1, P_2 and P_3, defined by the system of equations shown below.

$$P_1: \quad x + 2y + 2z = 4$$
$$P_2: \quad 2x + y - 2z = 5$$
$$P_3: 3x + 2y - 2z = 8$$

(i) Using the information given in part (a)(ii), show that the points $A(2, 1, 0)$ and $B(0, 3, -1)$ are common to all three planes.

(2 marks)

(ii) Show that P_1 and P_2 are perpendicular.

(2 marks)

(c) Figure 12 shows the point $C(0, 6, 2)$ on P_3, the point $D(12, -4, 0)$ on P_1, and the line l through C and D intersecting P_2 at the point E.

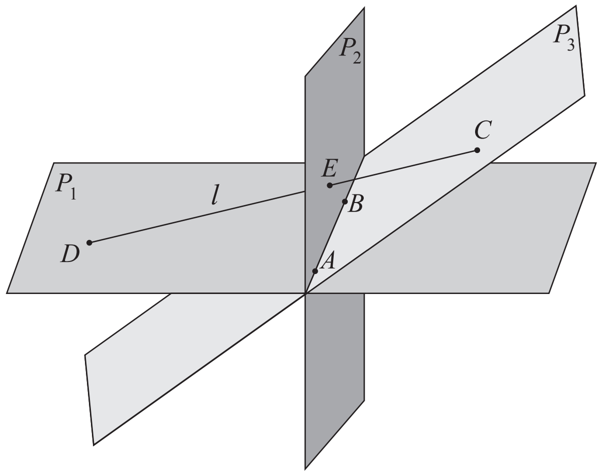


Figure 12

(i) Find, in parametric form, the equation of l.

(2 marks)

(ii) Find the coordinates of E.

(2 marks)

(iii) Find the distance from E to P_1.

(2 marks)

(d) The equations of P_1 and P_3 are used to model two hillsides that meet at a river, as shown in Figure 13.

$$P_1: \quad x+2y+2z=4$$
$$P_3: 3x+2y-2z=8$$

The river is modelled by the line where the two planes meet. A straight bridge, modelled by l, connects $C(0, 6, 2)$ to $D(12, -4, 0)$.

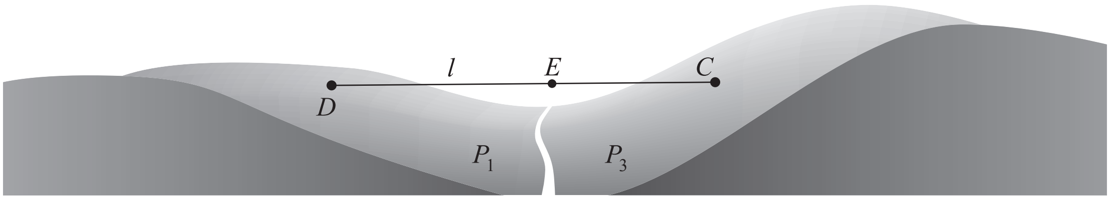


Figure 13

The point E, on the bridge, must be at least 1 unit from P_1 and at least 1 unit from P_3.

Does the model satisfy this condition? Show your calculations.

(2 marks)

QUESTION 13 (15 marks)

(a) (i) Solve $z^5 = -1$. Write your solutions in polar form.

(3 marks)

(ii) Draw the solutions on the Argand diagram in Figure 14, labelling each solution in an anticlockwise direction from z_1 to z_5, where z_1 is the solution with the smallest positive argument.

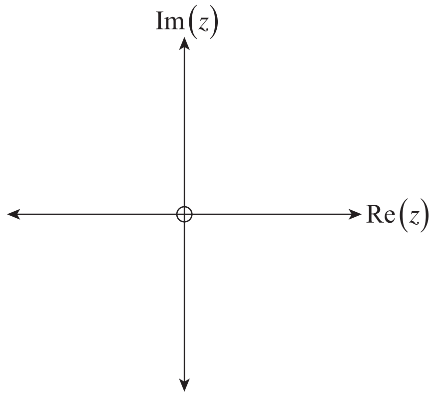


Figure 14

(2 marks)

Join your solutions labelled z_1, z_2, z_3, z_4, and z_5 to form a pentagon.

(iii) Show that $|z_1 - z_5| = 2\sin\dfrac{\pi}{5}$.

(2 marks)

(iv) Show that the perimeter of the pentagon is $10\sin\frac{\pi}{5}$.

(1 mark)

(v) Show that the area of the pentagon is $\frac{5}{2}\sin\frac{2\pi}{5}$.

(2 marks)

Consider the solutions to $z^n = -1$ for integers $n \geq 3$.

(b) A polygon is obtained by plotting and joining the solutions of $z^n = -1$.

Let $P(n)$ be the perimeter of this polygon.

(i) Write down an expression for $P(n)$.

(1 mark)

(ii) State the shape of the polygon formed as $n \to \infty$.

(1 mark)

(iii) What exact value does $P(n)$ approach as $n \to \infty$?

(1 mark)

(c) Let $A(n)$ be the area of the polygon described in part (b).

(i) Show that $A(n) = \frac{n}{2}\sin\frac{2\pi}{n}$.

(1 mark)

(ii) What exact value does $A(n)$ approach as $n \to \infty$?

(1 mark)

QUESTION 14 (15 marks)

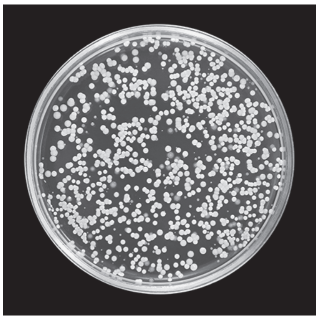
(a) In an experiment one type of bacterium, called alpha, was grown in a Petri dish.

The rate of change of the area in the Petri dish that is covered by alpha bacteria can be modelled by the differential equation

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}A\left(\frac{50-A}{50}\right)$$

where A is area in cm^2 and t is time in days.

Petri dish



Source: © Satirus | Shutterstock.com

(i) At $t = 0$, the area in the Petri dish that is covered by alpha bacteria is $1\ \text{cm}^2$.

On the slope field in Figure 15, draw the solution curve.

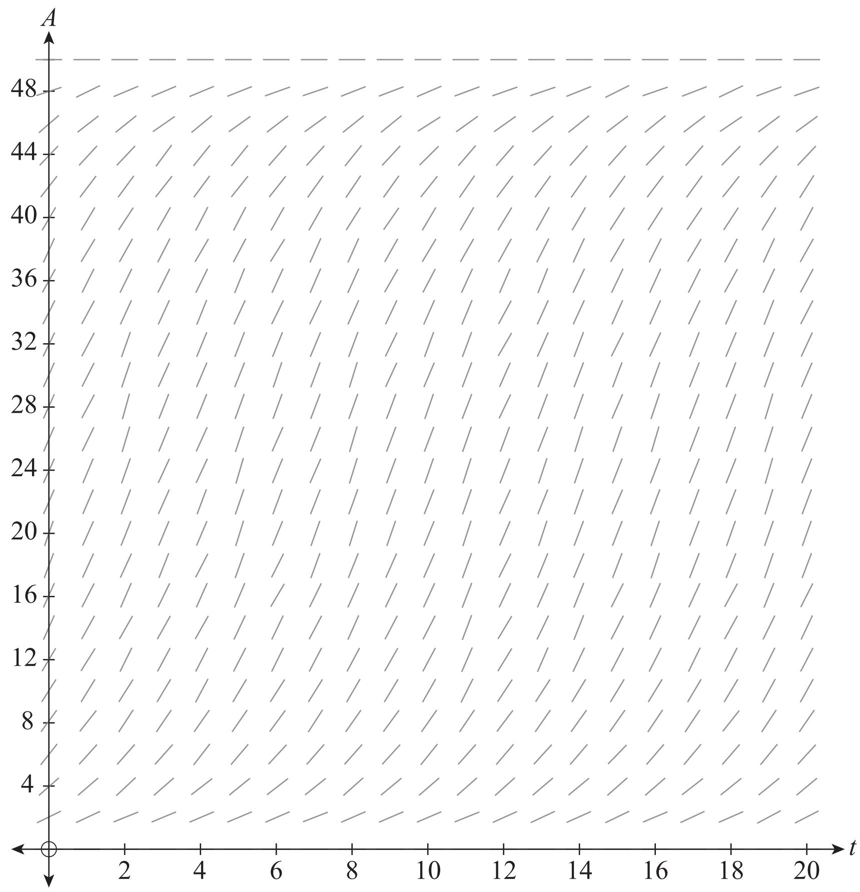


Figure 15

(3 marks)

(ii) Show that $\dfrac{50}{A(50-A)} = \dfrac{1}{A} + \dfrac{1}{50-A}$.

(1 mark)

(iii) Use integration to solve the differential equation

$$\frac{dA}{dt} = \frac{1}{2}A\left(\frac{50 - A}{50}\right)$$

with initial condition $A(0) = 1$, and show that the area covered by alpha bacteria can be modelled by

$$A = \frac{50}{1 + 49e^{-0.5t}}.$$

(5 marks)

(iv) State the maximum area that is available for bacterial growth.

(1 mark)

(b) When $t = 1$, a different bacterium, called beta, was accidentally introduced to the Petri dish. Beta bacteria grow more quickly than alpha bacteria. The area, B, in the Petri dish that is covered by beta bacteria can be modelled by

$$B = \frac{50}{1 + 79e^{-5d}}$$

where d is time in days after the beta bacterium was introduced.

Figure 16 shows the graph for the area in the Petri dish covered by beta bacteria d days after the beta bacterium was introduced.

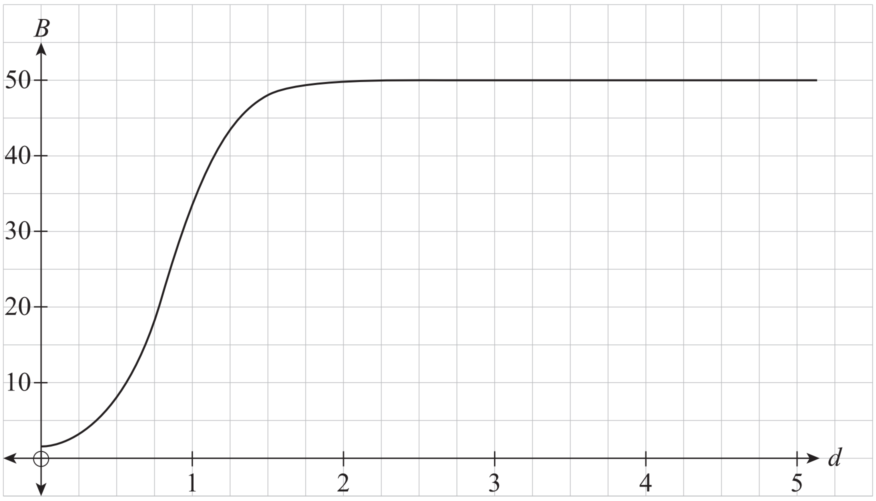


Figure 16

(i) Find d and B when the rate of growth of beta bacteria is at its greatest.

(2 marks)

(ii) Find the area of the Petri dish covered by alpha bacteria when the rate of growth of beta bacteria is at its greatest.

(2 marks)

(iii) When the Petri dish is completely covered by bacteria, which bacteria is likely to cover more area of the Petri dish?

(1 mark)

QUESTION 15 (15 marks)

(a) Consider the equation $x^3 + x^2 + x + 1 = 0$.

(i) Show that $x = -1$ is a root of this equation.

(1 mark)

(ii) Show that there are no other real roots.

(1 mark)

(b) Let $g(x) = \sqrt{x}$ and $h(x) = x^3 + x^2 + x + 1$.

(i) Find the composite function $g(h(x))$.

(1 mark)

(ii) State the domain of $g(h(x))$.

(1 mark)

(c) Consider $f(x)=\sqrt{x^3+x^2+x+1}$.

(i) Draw the graph of $y=f(x)$ on the axes in Figure 17.

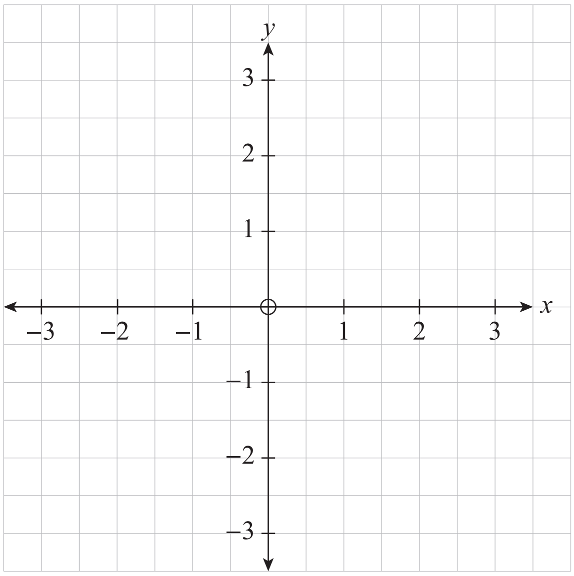


Figure 17

(2 marks)

(ii) Explain why $f(x)$ has an inverse, $f^{-1}(x)$.

(1 mark)

(iii) Using your answer to part (c)(i), sketch the graph of $f^{-1}(x)$ on the axes in Figure 18.

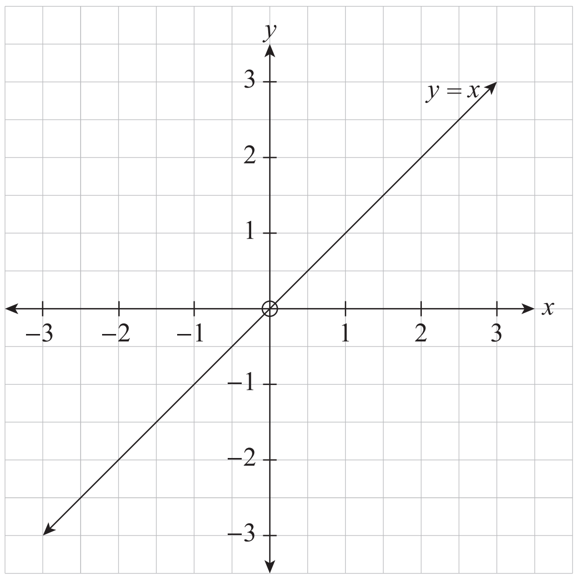


Figure 18

(2 marks)

(iv) (1) Find $f(1)$.

(1 mark)

(2) Find $f^{-1}(2)$.

(1 mark)

(d) If $y = f^{-1}(x)$, then $x = f(y)$.

(i) Use implicit differentiation to show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(f^{-1}(x)\right) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{f'(y)}$.

(1 mark)

(ii) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(f^{-1}(x)\right)$ at $x = 2$.

(3 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. Question 14(a)(iii) continued).