Question 5 (7 marks)

(a) Use mathematical induction to prove that for any positive integer n

$$1+\frac{1}{x}+\frac{1}{x^2}+\ldots+\frac{1}{x^n}=\frac{x^{n+1}-1}{x^n(x-1)}, \quad \text{if } x\neq 0, x\neq 1.$$

P_n is: $\displaystyle\sum_{i=0}^{n}\frac{1}{x^i}=\frac{x^{n+1}-1}{x^n(x-1)}$ if $x\neq 0, x\neq 1$ for any $n\in\mathbb{Z}^+$

If $n=1$, P_1 is: $1+\dfrac{1}{x}=\dfrac{x^2-1}{x(x-1)}$

$$\text{RHS}=\frac{(x+1)\cancel{(x-1)}}{x\cancel{(x-1)}}=1+\frac{1}{x}=\text{LHS} \qquad \therefore P_1 \text{ is true}$$

If P_k is true, then P_k is $\displaystyle\sum_{i=0}^{k}\frac{1}{x^i}=\frac{x^{k+1}-1}{x^k(x-1)}$

P_{k+1} should be: $\displaystyle\sum_{i=0}^{k+1}\frac{1}{x^i}=\frac{x^{k+2}-1}{x^{k+1}(x-1)}$

Now $\displaystyle\sum_{i=0}^{k+1}\frac{1}{x^i}=\sum_{i=0}^{k}\frac{1}{x^i}+\frac{1}{x^{k+1}}$

$$=\frac{x^{k+1}-1}{x^k(x-1)}+\frac{1}{x^{k+1}} \quad \text{using } P_k$$

$$=\frac{x}{x}\cdot\frac{x^{k+1}-1}{x^k(x-1)}+\frac{1}{x^{k+1}}\cdot\frac{(x-1)}{(x-1)}$$

$$=\frac{x^{k+2}-\cancel{x}+\cancel{x}-1}{x^{k+1}(x-1)}$$

$$=\frac{x^{k+2}-1}{x^{k+1}(x-1)} \qquad \therefore P_{k+1} \text{ is true whenever } P_k \text{ is true}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true, P_n is true for all $n\in\mathbb{Z}^+$ by the principle of mathematical induction.

(5 marks)

(b) Hence show that, for any positive integer n:

$$1+\frac{1}{11}+\frac{1}{11^2}+\ldots+\frac{1}{11^n}<1.1.$$

Substituting $x = 11$ in P_n gives:

$$1 + \frac{1}{11} + \frac{1}{11^2} + \ldots + \frac{1}{11^n} = \frac{11^{n+1} - 1}{11^n \cdot 10}$$

$$< \frac{11^{n+1}}{11^n \cdot 10}$$

$$= \frac{11}{10}$$

$$= 1{\cdot}1$$

(2 marks)