Question 3 (6 marks)

Let $A=\begin{bmatrix} \frac{1}{9} & 0 \\ 0 & 2 \end{bmatrix}$.

(a) Use mathematical induction to prove that $A^n=\begin{bmatrix} \left(\frac{1}{3}\right)^{2n} & 0 \\ 0 & 2^n \end{bmatrix}$ for all positive integers n.

P_n is : $\begin{bmatrix} 1/9 & 0 \\ 0 & 2 \end{bmatrix}^n = \begin{bmatrix} (1/3)^{2n} & 0 \\ 0 & 2^n \end{bmatrix}$ for all $n \in \mathbb{Z}^+$

If $n=1$, P_1 is : $\begin{bmatrix} 1/9 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/9 & 0 \\ 0 & 2 \end{bmatrix}$ $\therefore P_1$ is true

If P_k is true, then P_k is : $\begin{bmatrix} 1/9 & 0 \\ 0 & 2 \end{bmatrix}^k = \begin{bmatrix} (1/3)^{2k} & 0 \\ 0 & 2^k \end{bmatrix}$

P_{k+1} should be : $\begin{bmatrix} 1/9 & 0 \\ 0 & 2 \end{bmatrix}^{k+1} = \begin{bmatrix} (1/3)^{2k+2} & 0 \\ 0 & 2^{k+1} \end{bmatrix}$

Now $\begin{bmatrix} 1/9 & 0 \\ 0 & 2 \end{bmatrix}^{k+1} = \begin{bmatrix} 1/9 & 0 \\ 0 & 2 \end{bmatrix}^k . \begin{bmatrix} 1/9 & 0 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} (1/3)^{2k} & 0 \\ 0 & 2^k \end{bmatrix} . \begin{bmatrix} (1/3)^2 & 0 \\ 0 & 2 \end{bmatrix}$ using P_k

$= \begin{bmatrix} (1/3)^{2k+2} & 0 \\ 0 & 2^{k+1} \end{bmatrix}$ $\therefore P_{k+1}$ is true whenever P_k is true

Since P_1 is true, and P_{k+1} is true whenever P_k is true, P_n is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

(5 marks)

(b) Using part (a), find the positive integer n such that $A^n\begin{bmatrix} 0 \\ 8 \end{bmatrix}=\begin{bmatrix} 0 \\ 2^{2021} \end{bmatrix}$.

$\begin{bmatrix} (1/3)^{2n} & 0 \\ 0 & 2^n \end{bmatrix} . \begin{bmatrix} 0 \\ 2^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2^{2021} \end{bmatrix}$ using P_n

$2^{n+3} = 2^{2021} \Rightarrow n = 2018$

(1 mark)