## Question 5 (7 marks)

(a) Use mathematical induction to prove that $7^n + 2$ is divisible by 3 for all positive integers $n$.

$P_n$ is : $3 \mid 7^n + 2$ for all $n \in \mathbb{Z}^+$

If $n=1$, $P_1$ is : $3 \mid 7 + 2$

ie. $3 \mid 9$ $\therefore P_1$ is true

If $P_k$ is true, then $P_k$ is: $7^k + 2 = 3A$ for some $A \in \mathbb{Z}$

i.e. $7^k = 3A - 2$

$P_{k+1}$ should be : $7^{k+1} + 2 = 3B$ for some $B \in \mathbb{Z}$

Now $7^{k+1} + 2 = 7 \times 7^k + 2$

$= 7(3A - 2) + 2$ using $P_k$

$= 21A - 14 + 2$

$= 21A - 12$

$= 3(7A - 4)$

$= 3B$ where $B \in \mathbb{Z}$ since $A \in \mathbb{Z}$

$\therefore P_{k+1}$ is true whenever $P_k$ is true

Since $P_1$ is true, and $P_{k+1}$ is true whenever $P_k$ is true, $P_n$ is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction

(5 marks)

(b) Hence show that $7^n - 1$ is divisible by $3$ for all positive integers $n$.

$7^n - 1 = 7^n + 2 - 3$

$= 3(c - 1)$ for some $c \in \mathbb{Z}$ $\quad \therefore 3 \mid 7^n - 1$

(1 mark)

(c) Use parts (a) and (b) to show that $7^{2n} + 7^n - 2$ is divisible by $9$ for all positive integers $n$.

$7^{2n} + 7^n - 2 = (7^n - 1)(7^n + 2)$

$\therefore 9 \mid 7^{2n} + 7^n - 2$ since $3 \mid 7^n - 1$ and $3 \mid 7^n + 2$

(1 mark)