Question 5 (8 marks)

(a) Use mathematical induction to prove that

$$4+4^2+4^3+\ldots+4^n=\frac{4}{3}\left(4^n-1\right)$$

where n is a positive integer.

P_n is : $\displaystyle\sum_{i=1}^{n} 4^i = \frac{4}{3}(4^n-1)$

If $n=1$, P_1 is $4 = \frac{4}{3}(4-1)$

ie. $4 = \frac{4}{\cancel{3}} \times \cancel{3}$ $\therefore P_1$ is true

If P_k is true, then P_k is : $\displaystyle\sum_{i=1}^{k} 4^i = \frac{4}{3}(4^k-1)$

P_{k+1} should be : $\displaystyle\sum_{i=1}^{k+1} 4^i = \frac{4}{3}(4^{k+1}-1)$

Now $\displaystyle\sum_{i=1}^{k+1} 4^i = \sum_{i=1}^{k} 4^i + 4^{k+1}$

$= \frac{4}{3}(4^k-1) + \frac{3\times4\times4^k}{3}$ using P_k

$= \frac{4}{3}(4^k-1+3\times4^k)$

$= \frac{4}{3}(4\times4^k-1)$

$= \frac{4}{3}(4^{k+1}-1)$ $\therefore P_{k+1}$ is true whenever P_k is true

Since P_1 is true, and P_{k+1} is true whenever P_k is true, P_n is true for all $n\in\mathbb{Z}^+$ by the principle of mathematical induction.

(6 marks)

(b) Hence show that $3+15+63+\ldots+16777215=22369608$.

$$3+15+63+\ldots+16777215 = \sum_{i=1}^{12} 4^i - 1$$

$$= \frac{4}{3}(4^{12}-1) - 12$$

$$= 22369608$$

(2 marks)