QUESTION 5 (8 marks)

(a) Use mathematical induction to prove that:

$$\frac{1}{4\times1^2-1}+\frac{1}{4\times2^2-1}+\ldots+\frac{1}{4\times n^2-1}=\frac{n}{2n+1}, \text{ where } n \text{ is a positive integer.}$$

P_n is : $\displaystyle\sum_{i=1}^{n} \frac{1}{4i^2-1} = \frac{n}{2n+1}$ for all $n \in \mathbb{Z}^+$

If $n=1$, P_1 is : $\dfrac{1}{4\times1^2-1} = \dfrac{1}{2\times1+1}$

ie, $\dfrac{1}{3} = \dfrac{1}{3}$ $\therefore P_1$ is true

If P_k is true, then P_k is : $\displaystyle\sum_{i=1}^{k} \frac{1}{4i^2-1} = \frac{k}{2k+1}$

P_{k+1} should be : $\displaystyle\sum_{i=1}^{k+1} \frac{1}{4i^2-1} = \frac{k+1}{2k+3}$

Now $\displaystyle\sum_{i=1}^{k+1} \frac{1}{4i^2-1} = \sum_{i=1}^{k} \frac{1}{4i^2-1} + \frac{1}{4(k+1)^2-1}$

$= \dfrac{k}{2k+1} + \dfrac{1}{4(k+1)^2-1}$ using P_k

$= \dfrac{k}{(2k+1)}\dfrac{(2k+3)}{(2k+3)} + \dfrac{1}{(2k+1)(2k+3)}$

$= \dfrac{2k^2+3k+1}{(2k+1)(2k+3)}$

$= \dfrac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}$

$= \dfrac{k+1}{2k+3}$ $\therefore P_{k+1}$ is true whenever P_k is true

Since P_1 is true, and P_{k+1} is true whenever P_k is true, P_n is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

(6 marks)

(b) Hence find, in simplest rational form, the value of $\frac{1}{15}+\frac{1}{35}+\ldots+\frac{1}{399}$.

$$\frac{1}{15}+\frac{1}{35}+\ldots+\frac{1}{399} = \sum_{i=2}^{10} \frac{1}{4i^2-1}$$

$$= \sum_{i=1}^{10} \frac{1}{4i^2-1} - \frac{1}{3}$$

$$= \frac{10}{21} - \frac{7}{21} \quad \text{using } P_{10}$$

$$= \frac{1}{7}$$

(2 marks)