South Australian
Certificate of Education

1

Mathematical Methods
2022

Question booklet 1

- Questions 1 to 6 (56 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on page 16 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

Examination information

Materials
- Question booklet 1
- Question booklet 2
- Formula sheet
- SACE registration number label

Instructions
- Show appropriate working and steps of logic in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams and graphical representations

Total time: 130 minutes
Total marks: 100

Attach your SACE registration number label here	**Graphics calculator** 1. Brand _____ Model _____ 2. Brand _____ Model _____

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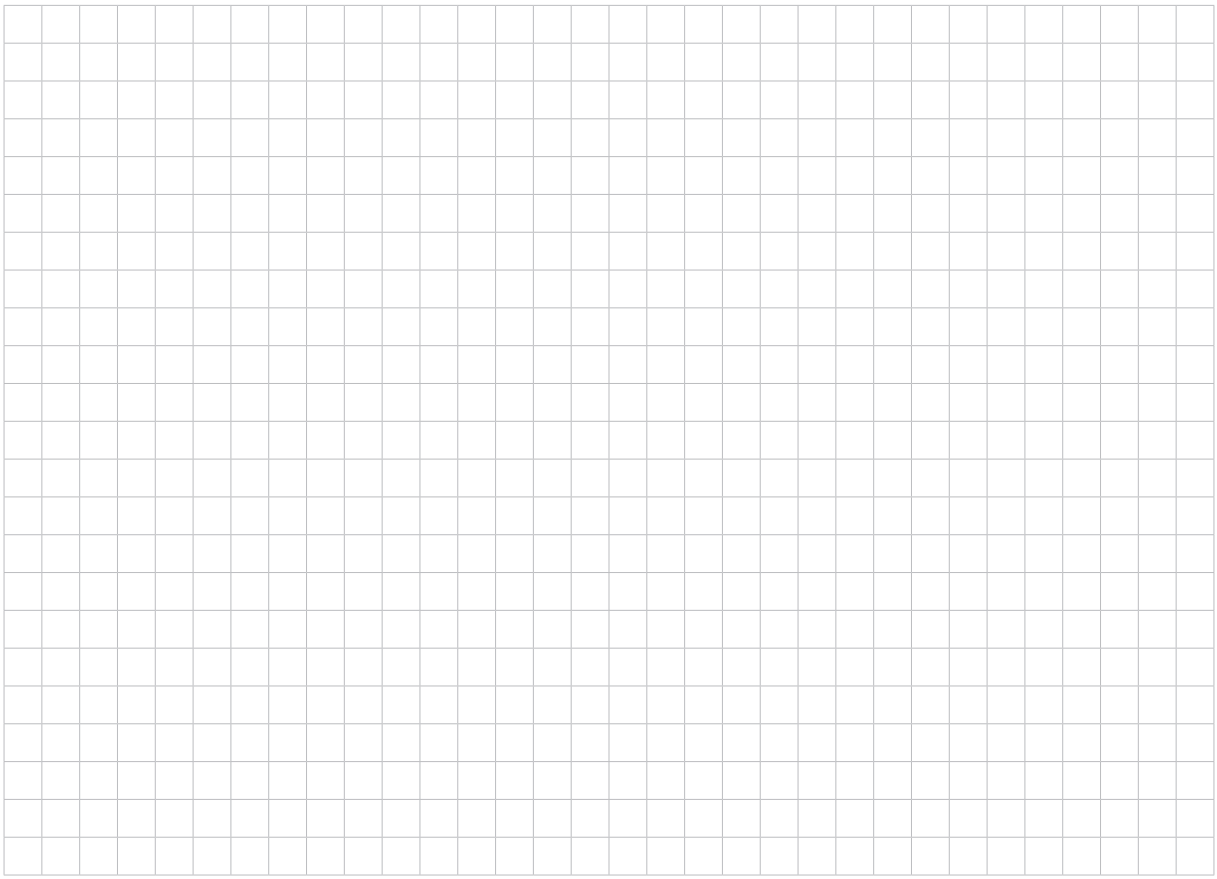
Question 1 (11 marks)

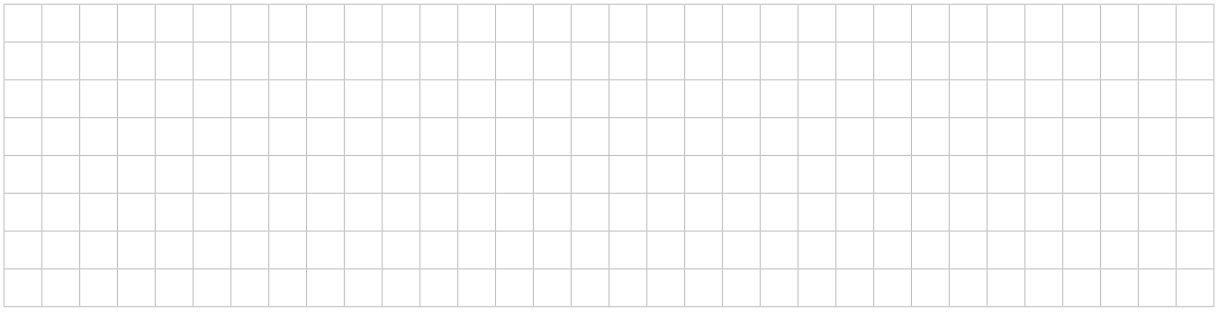
(a) *Using first principles*, find $f'(x)$ if $f(x) = x^2 - 3$.



(3 marks)

(b) Determine $\frac{dy}{dx}$ for the following two functions. You do not need to simplify your answers.

(i) $y = 3x^4 + (7 + \sin x)^2$



(3 marks)

(ii) $y = \dfrac{\ln\left(x^5 + 1\right)}{e^x}$

(3 marks)

(c) Determine $\int\left(\dfrac{1}{x-3} + 4\right)\mathrm{d}x$, where $x > 3$.

(2 marks)

Question 2 (7 marks)

The leading digit of a number is defined as the left-most non-zero digit. In the numbers given below, for example, 7 is the leading digit.

$$712 \qquad 74.2 \qquad 0.00798$$

Let X represent the leading digit of a naturally occurring number. Benford's law states that the leading digits in large sets of naturally occurring numbers are distributed as follows:

x	1	2	3	4	5	6	7	8	9
$\Pr(X = x)$	0.301	0.176	0.125	0.097	0.079	0.067	0.058	0.051	0.046

(a) What property of the values in the table above mean that X is **not** uniformly distributed?

(1 mark)

(b) Determine the probability of randomly selecting a naturally occurring number with a leading digit of 6 or greater.

(1 mark)

(c) Identify which of the following provides a correct calculation for the mean of the leading digit of a naturally occurring number, μ_X. Tick the appropriate box.

☐ $\mu_X = 1^2(0.301) + 2^2(0.176) + 3^2(0.125) + 4^2(0.097) + 5^2(0.079) + 6^2(0.067) + 7^2(0.058) + 8^2(0.051) + 9^2(0.046)$

☐ $\mu_X = 1(0.301) + 2(0.176) + 3(0.125) + 4(0.097) + 5(0.079) + 6(0.067) + 7(0.058) + 8(0.051) + 9(0.046)$

☐ $\mu_X = \dfrac{0.301 + 0.176 + 0.125 + 0.097 + 0.079 + 0.067 + 0.058 + 0.051 + 0.046}{9}$

(1 mark)

(d) Benford's law can be used to detect fraudulent financial records. Further investigation of a company's financial records is undertaken if a calculated 95% confidence interval supports the claim that the mean of the leading digit of the financial records is greater than 3.441.

Consider a random sample of 250 numbers from a company's records. The mean of the leading digit in this sample is $\bar{x} = 3.816$.

(i) Calculate a 95% confidence interval for the mean of the leading digit of the company's financial records, using the random sample described above. Assume that the standard deviation is $\sigma_X = 2.462$.

(2 marks)

(ii) Does this 95% confidence interval suggest that further investigation should be undertaken? Justify your answer.

(2 marks)

Question 3 (7 marks)

Consider the function $f(x)$ for $x \geq 0$ and $x \neq 4$. The graph of $y = f(x)$ is shown in Figure 1.

Points A and C are the function's only stationary points, with x-coordinates of $x = 1$ and $x = 3$ respectively. Point B is the function's only inflection point, with an x-coordinate of $x = 2$. The function also has a vertical asymptote with equation $x = 4$.

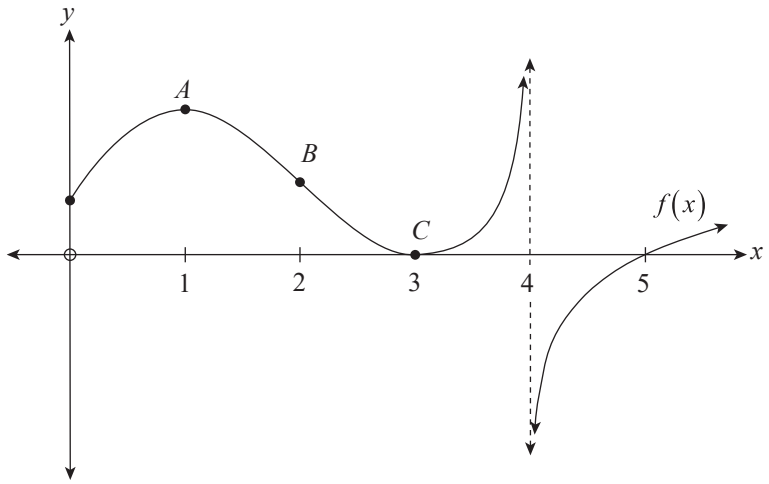


Figure 1

(a) (i) Complete the sign diagram below for $f'(x)$.

(2 marks)

(ii) Complete the sign diagram below for $f''(x)$.

(2 marks)

(b) *On the axes in Figure 2*, sketch a possible graph of $y = f'(x)$.

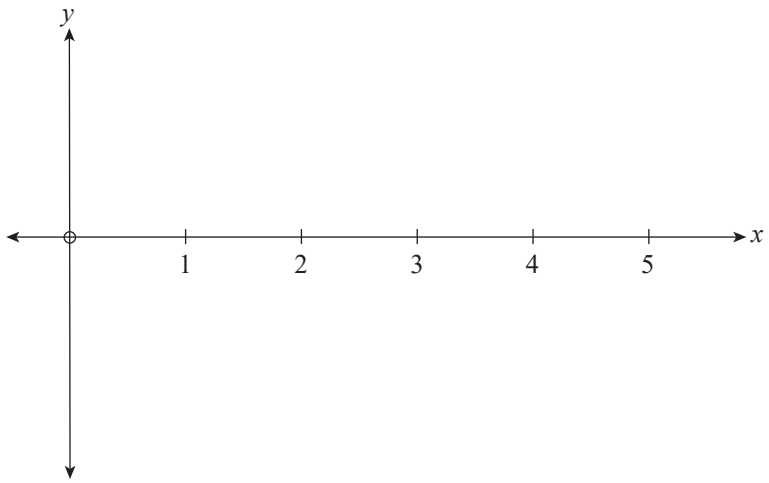


Figure 2

(3 marks)

Question 4 (9 marks)

The fat content in grams, X, per carton of Ted's Oat Milk can be modelled using a normal distribution with a mean of $\mu_X = 2.7\,\text{g}$ and a standard deviation of $\sigma_X = 0.35\,\text{g}$.

(a) Determine the probability that a carton of Ted's Oat Milk will contain:

(i) at least $2.8\,\text{g}$ of fat

(1 mark)

(ii) between $1\,\text{g}$ and $3\,\text{g}$ of fat.

(1 mark)

(b) Ted's Oat Milk is sold in packs of six cartons. Let S_6 represent the distribution of fat content in one six-pack of Ted's Oat Milk.

(i) Explain why S_6 can be modelled by a normal distribution.

(1 mark)

(ii) Show that the distribution of S_6 has a mean of $\mu_{S_6} = 16.2\,\text{g}$ and a standard deviation of $\sigma_{S_6} = 0.857\,\text{g}$ (correct to three significant figures).

(2 marks)

(iii) Hence, determine the probability that a six-pack of Ted's Oat Milk will contain less than 18 g of fat.

(1 mark)

(c) Find the value of k if $\Pr(Z < k) = 0.99$, given that $Z \sim N(0,1)$.

(1 mark)

(d) The label on the side of a six-pack of Ted's Oat Milk states that:

'Each pack of 6 contains less than 18 g of fat.'

The company would like this statement to be true for at least 99% of Ted's Oat Milk six-packs and it is considering improvements to the manufacturing process. Although it cannot change the mean fat content of a six-pack through these improvements, the standard deviation can be reduced.

Source: © adapted from Photeuphoria | Dreamstime.com

Using your answer to part (c), determine the largest possible value of σ_{S_6} that will result in at least 99% of six-packs containing less than 18 g of fat. Assume that $\mu_{S_6} = 16.2$ g.

(2 marks)

Question 5 (8 marks)

Consider the function $f(x) = \cos\left(\frac{1}{5}x^2 - 1\right)$ for $0 \le x \le \pi$. Figure 3 shows the graph of $y = f(x)$, which has two stationary points: one located at $x = 0$, and the other located at $x = a$, where $0 < a < \pi$.

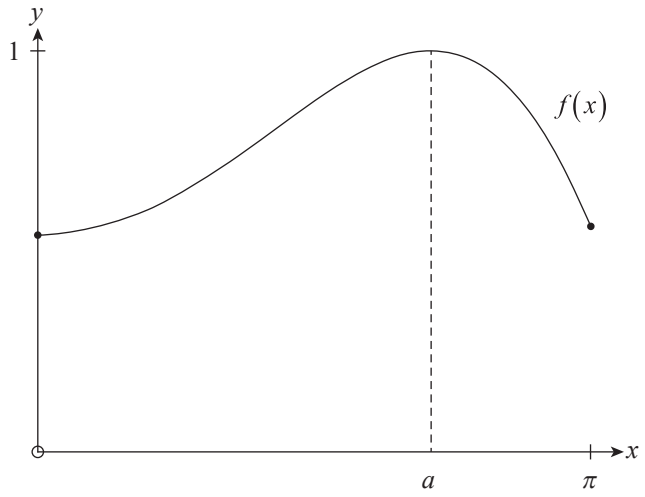


Figure 3

(a) State the nature of the stationary point located at $x = a$.

(1 mark)

(b) (i) Determine $f'(x)$.

(2 marks)

(ii) Hence, using an algebraic approach, determine the exact value of a.

(3 marks)

(iii) State the interval of x for which $f(x)$ is increasing.

(1 mark)

(iv) Given that $f(x)$ has no inflection points to the right of the stationary point located at $x = a$, state the x-coordinate(s) where $f(x)$ is decreasing the most rapidly.

(1 mark)

Question 6 (14 marks)

The rate at which medical samples arrive at a laboratory t hours after 8.00 am on a particular day can be modelled using the function

$$m(t) = 500t\sqrt{t}\,e^{-0.9t} \text{ for } 0 \le t \le 9,$$

where the rate at which medical samples arrive at the laboratory, $m(t)$, is measured in samples per hour.

Source: adapted from Belova59 | pixabay.com

(a) (i) Show that $m'(t) = 750\sqrt{t}\,e^{-0.9t} - 450t\sqrt{t}\,e^{-0.9t}$.

(2 marks)

(ii) Hence, using an algebraic approach, show that the rate at which medical samples arrive at the laboratory is maximised at $t = \frac{5}{3}$, according to the model.

(3 marks)

Let the total number of medical samples that arrive at the laboratory between 8.00 am and 5.00 pm on a particular day be N.

(b) An overestimate for the value of N can be calculated using three rectangles of equal width. These three rectangles, along with the graph of $y = m(t)$ where $0 \leq t \leq 9$, are shown in Figure 4.

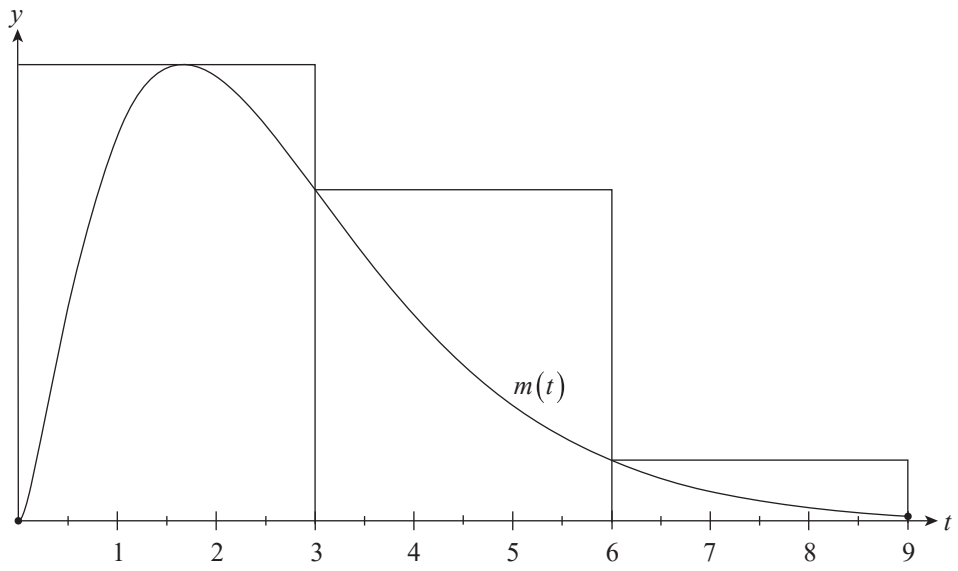


Figure 4

(i) Calculate the value of this overestimate.

(2 marks)

(ii) If an overestimate of N was calculated using an increasing number of rectangles of equal width, it would approach the true value of N.

Calculate this value of N, correct to the nearest integer.

(1 mark)

When a medical sample arrives at the laboratory, it is placed in a queue before being processed. The medical samples are processed at a constant rate of 150 samples per hour.

Let the function $p(t) = 150$ represent the rate at which medical samples are processed.

Figure 5 shows the graphs of $y = m(t)$ and $y = p(t)$, where $0 \leq t \leq 9$. The graphs of $y = m(t)$ and $y = p(t)$ intersect at $t = 0.670$ and $t = 3.36$ (correct to three significant figures).

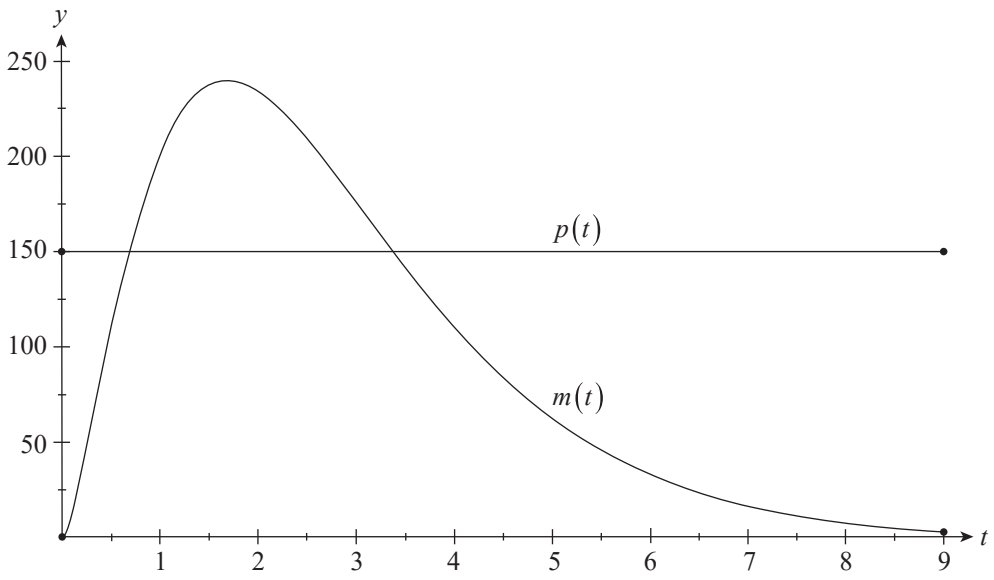


Figure 5

(c) At 8.00 am $(t = 0)$, the laboratory had 600 medical tests in the queue to be processed.

(i) Explain why the number of items in the queue is decreasing until 0.670 hours after 8.00 am.

(1 mark)

(ii) During the time that the number of items in the queue was increasing, the queue increased by K medical samples.

(1) Write an integral expression that could be used to calculate K.

(2 marks)

(2) Hence, or otherwise, determine the value of K, correct to the nearest integer.

(1 mark)

(iii) Determine the number of medical samples in the queue at 5.00 pm.

(2 marks)

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 6(a)(ii) continued).

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Mathematical Methods
2022

Question booklet 2

- Questions 7 to 10 (44 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on pages 6 and 15 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

2

Copy the information from your SACE label here

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2. Brand _____

Model _____

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Question 7 (9 marks)

The graph of $y = x \ln x - x$ where $x > 0$ is shown in Figure 6.

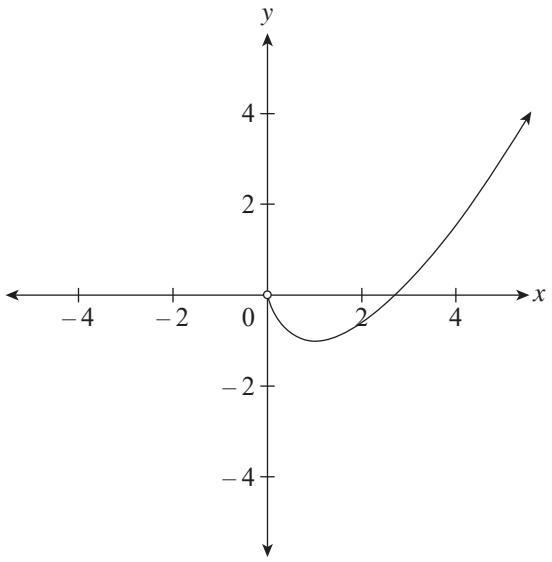

Figure 6

(a) (i) Show that $\frac{dy}{dx} = \ln x$.

(1 mark)

(ii) Hence, using an algebraic approach, determine the y-coordinate of the stationary point of $y = x \ln x - x$.

(2 marks)

Consider the family of functions of the form $y = x\ln x - x + (\ln k)x$, where $x > 0$ and k is a positive constant. Each function in this family has a stationary point at A, as shown in Figure 7.

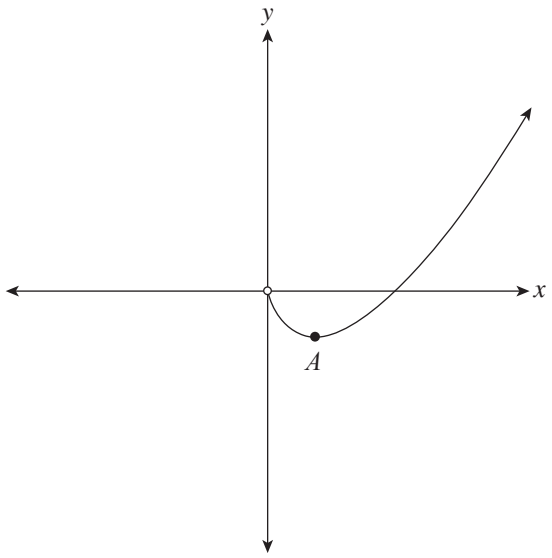

Figure 7

(b) Complete the table below by finding the y-coordinates of A for $y = x\ln x - x + (\ln k)x$ where $k = 1, 2, 3$, and 4.

k	*Function*	*y-coordinate of A*
1	$x\ln x - x$	
2	$x\ln x - x + (\ln 2)x$	
3	$x\ln x - x + (\ln 3)x$	
4	$x\ln x - x + (\ln 4)x$	

(1 mark)

(c) Hence, make a conjecture for the y-coordinate of A, for any value of k.

(1 mark)

(d) Prove or disprove your conjecture.

(4 marks)

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 7(a)(ii) continued).

Question 8 (16 marks)

A company produces a type of fruit-filled chocolate called Choc-pricots. They are produced on an automated assembly line that can result in imperfect Choc-pricots.

Source: © Hendrik Schmidt 2015 | Alamy.com

(a) The probability is 0.15 that a Choc-pricot randomly selected off the assembly line is imperfect.

Use the binomial distribution to determine the probability that, in a random sample of 80 Choc-pricots:

(i) *exactly* 6 Choc-pricots will be imperfect.

(1 mark)

(ii) *more than* 20 Choc-pricots will be imperfect.

(2 marks)

Question 8 continues on page 8.

The manager of the assembly line can adjust the rate of production of Choc-pricots, resulting in a change in the probability that a randomly selected Choc-pricot is imperfect. This probability can be modelled by the function

$$p(x) = \sin\left(0.00392x^2\right) + e^{-x}, \text{ for } 0 < x \leq 20,$$

where $p(x)$ represents the probability that a randomly selected Choc-pricot is imperfect, and x represents the rate of production measured in Choc-pricots per second.

(b) *On the axes in Figure 8*, sketch the graph of $y = p(x)$.

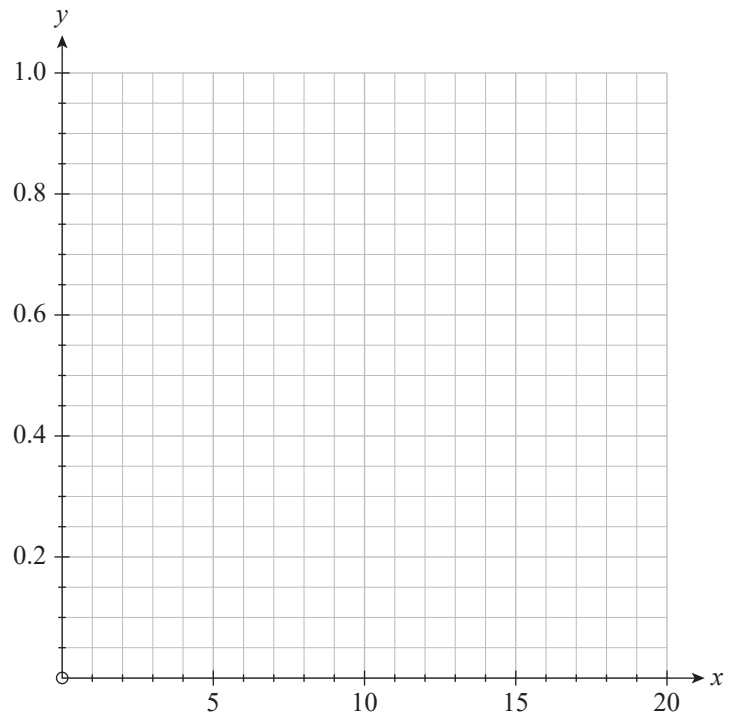

Figure 8

(3 marks)

(c) Using the function $p(x)$, determine the possible rate(s) of production that resulted in the given probability of 0.15 stated in part (a).

(2 marks)

(d) The minimum probability that a randomly selected Choc-pricot is imperfect, according to the function $p(x)$, is 0.0781 (correct to three significant figures).

State the corresponding rate of production that would result in this probability.

(1 mark)

(e) The manager chooses a new production rate for the assembly line. They claim that the production rate that they have chosen will result in the minimum probability that a randomly selected Choc-pricot is imperfect.

After running the assembly line at the new production rate for one day, a quality-control officer randomly selects 80 Choc-pricots and 6 are imperfect.

(i) Calculate $\hat{p}$, the proportion of Choc-pricots that are imperfect in the sample of 80 Choc-pricots manufactured at the new production rate.

(1 mark)

(ii) Assuming that a sample size of 80 is large enough for the central limit theorem to apply, determine a 90% confidence interval for the proportion of Choc-pricots that will be imperfect at the new production rate.

(2 marks)

(iii) Using the confidence interval that you calculated in part (e)(ii), can it be concluded that the manager's claim is false with 90% confidence? Justify your answer.

(2 marks)

(iv) Using the confidence interval that you calculated in part (e)(ii) and $p(x)$, determine with 90% confidence the upper and lower bounds of the new production rate.

(2 marks)

Question 9 (11 marks)

Let $f(x) = \sqrt{8 - x^3}$ for $x < 2$.

(a) Show that $f'(x) = \dfrac{-3x^2}{2\sqrt{8 - x^3}}$ for $x < 2$.

(1 mark)

The graph of $y = f(x)$ is shown in Figure 9, along with the normal to the graph at $x = -1$.

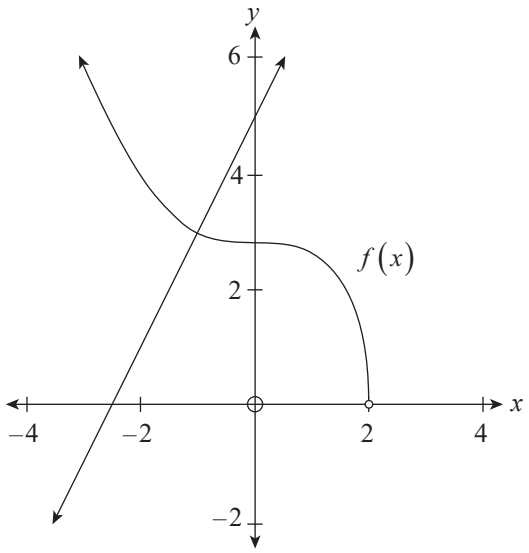

Figure 9

(b) (i) Show that the normal to the graph of $y = f(x)$ at $x = -1$ has the equation $2x - y = -5$.

(3 marks)

(ii) State the x-intercept of the normal $2x-y=-5$.

(1 mark)

(c) Consider the normal to the graph of $y=f(x)$ at $x=a$, where $a<2$.

(i) Show that the x-intercept of this normal is $x=a-\frac{3}{2}a^2$.

(4 marks)

(ii) Hence, using an algebraic approach, find the value of a such that the x-intercept of this normal is maximised.

(2 marks)

Question 10 (8 marks)

The service life of an item is defined as an item's total life in use from the point of sale to the point of discard. A probability density function often used to model the service life of items is of the form

$$p_k(x) = \frac{k}{2}\left(\frac{x}{2}\right)^{k-1} e^{-\left(\frac{x}{2}\right)^k}$$

where x is measured in years for $x \geq 0$, and k is a constant such that $k \geq 1$. The value of k is determined by the failure rate of the item over time.

(a) Consider the case where $k = 2$, resulting in the probability density function

$$p_2(x) = \frac{x}{2} e^{-\left(\frac{x}{2}\right)^2}.$$

The function $p_2(x)$ can be used to model the service life of Product A.

(i) Using definite integrals, determine the probability that a randomly selected Product A will have a service life of 1.5 years or less.

(1 mark)

(ii) An approximate measure of the anticipated service life for Product A is m, the median service life of the item, which can be calculated using the equation:

$$\int_0^m p_2(x)\,\mathrm{d}x = 0.5.$$

Using your answer to part (a)(i), explain why the median service life of Product A is greater than 1.5 years.

(1 mark)

(b) (i) If $y = -e^{-\left(\frac{x}{2}\right)^k}$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{2}\left(\frac{x}{2}\right)^{k-1} e^{-\left(\frac{x}{2}\right)^k}$.

(1 mark)

When calculating the median service life of an item, m, using the probability density function $p_k(x)$, the following equation can be used:

$$\int_0^m p_k(x)\,\mathrm{d}x = 0.5.$$

(ii) Determine the *exact* value of k that would result in $p_k(x)$ having a median service life of 1.5 years.

(5 marks)

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You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 9(c)(i) continued).

MATHEMATICAL METHODS FORMULA SHEET

Properties of derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(f(x)\,g(x)\Big) = f'(x)g(x) + f(x)g'(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\big[g(x)\big]^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$$

Quadratic equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Discrete random variables

The mean or expected value of a discrete random variable is:

$$\mu_X = \sum xp(x),$$

where $p(x)$ is the probability function for achieving result x.

The standard deviation of a discrete random variable is:

$$\sigma_X = \sqrt{\sum \big[x - \mu_X\big]^2 p(x)},$$

where μ_X is the expected value and $p(x)$ is the probability function for achieving result x.

Bernoulli distribution

The mean of the Bernoulli distribution is p, and the standard deviation is:

$$\sqrt{p(1-p)}.$$

Binomial distribution

The mean of the binomial distribution is np, and the standard deviation is:

$$\sqrt{np(1-p)},$$

where p is the probability of success in a single Bernoulli trial and n is the number of trials.

The probability of k successes from n trials is:

$$\Pr(X = k) = C_k^n p^k (1-p)^{n-k},$$

where p is the probability of success in the single Bernoulli trial.

Population proportions

The sample proportion is $\hat{p} = \dfrac{X}{n}$,

where a sample of size n is chosen, and X is the number of elements with a given characteristic.

The distribution of a sample proportion has a mean of p and a standard deviation of

$$\sqrt{\frac{p(1-p)}{n}}.$$

The upper and lower limits of a confidence interval for the population proportion are:

$$\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},$$

where the value of z is determined by the confidence level required.

Continuous random variables

The mean or expected value of a continuous random variable is:

$$\mu_X = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x,$$

where $f(x)$ is the probability density function.

The standard deviation of a continuous random variable is:

$$\sigma_X = \sqrt{\int_{-\infty}^{\infty} \big[x - \mu_X\big]^2 f(x)\,\mathrm{d}x},$$

where $f(x)$ is the probability density function.

Normal distributions

The probability density function for the normal distribution with mean μ and standard deviation σ is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

All normal distributions can be transformed to the standard normal distribution with $\mu = 0$ and $\sigma = 1$ by:

$$Z = \frac{X - \mu}{\sigma}.$$

Sampling and confidence intervals

If $\bar{x}$ is the sample mean of a sufficiently large sample, and σ is the population standard deviation, then the upper and lower limits of the confidence interval for the population mean are:

$$\bar{x} - z\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + z\frac{\sigma}{\sqrt{n}},$$

where the value of z is determined by the confidence level required.