1

Government of South Australia

SACE Board of SA | South Australian Certificate of Education

Mathematical Methods
2017

Question Booklet 1

- **Part 1** (Questions 1 to 8) 66 marks
- Answer ***all*** questions in Part 1
- Write your answers in this question booklet
- You may write on page 20 if you need more space
- Allow approximately 90 minutes
- Approved calculators may be used — complete the box below

Examination information

Materials
- Question Booklet 1 (Part 1)
- Question Booklet 2 (Part 2)
- SACE registration number label

Reading time
- 10 minutes
- You may make notes on scribbling paper

Writing time
- 3 hours
- Show all working in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams

Total marks 139

Attach your SACE registration number label here

Graphics calculator

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Model ____________

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You may remove this page from the booklet by tearing along the perforations.

LIST OF MATHEMATICAL FORMULAE FOR USE IN STAGE 2 MATHEMATICAL METHODS

Properties of Derivatives

$$\frac{d}{dx}\Big(f(x)\,g(x)\Big) = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

$$\frac{d}{dx}f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$$

Quadratic Equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Discrete Random Variables

The mean or expected value of a discrete random variable is:

$$\mu_X = \sum x.p(x),$$

where $p(x)$ is the probability function for achieving result x.

The standard deviation of a discrete random variable is:

$$\sigma_X = \sqrt{\sum\left[x - \mu_X\right]^2 p(x)},$$

where μ_X is the expected value and $p(x)$ is the probability function for achieving result x.

Bernoulli Distribution

The mean of the Bernoulli distribution is p, and the standard deviation is:

$$\sqrt{p(1-p)}.$$

Binomial Distribution

The mean of the binomial distribution is np, and the standard deviation is:

$$\sqrt{np(1-p)},$$

where p is the probability of success in a single Bernoulli trial and n is the number of trials.

The probability of k successes from n trials is:

$$\Pr(X = k) = C_k^n p^k (1-p)^{n-k},$$

where p is the probability of success in the single Bernoulli trial.

Population Proportions

The sample proportion is $\hat{p} = \dfrac{X}{n}$,

where sample of size n is chosen, and X is the number of elements with a given characteristic.

The distribution of a sample proportion has a mean of p and a standard deviation of

$$\sqrt{\frac{p(1-p)}{n}}.$$

The upper and lower limits of a confidence interval for the population proportion are:

$$\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},$$

where the value of z is determined by the confidence level required.

Continuous Random Variables

The mean or expected value of a continuous random variable is:

$$\mu_X = \int_{-\infty}^{\infty} x\,f(x)\,dx,$$

where $f(x)$ is the probability density function.

The standard deviation of a continuous random variable is:

$$\sigma_X = \sqrt{\int_{-\infty}^{\infty}\left[x - \mu_X\right]^2 f(x)\,dx},$$

where $f(x)$ is the probability density function.

Normal Distributions

The probability density function for the normal distribution with the mean μ and the standard deviation σ is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

All normal distributions can be transformed to the standard normal distribution with $\mu = 0$ and $\sigma = 1$ by:

$$Z = \frac{X - \mu}{\sigma}.$$

Sampling and Confidence Intervals

If $\bar{x}$ is the sample mean and s the standard deviation of a suitably large sample, then the upper and lower limits of the confidence interval for the population mean are:

$$\bar{x} - z\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + z\frac{s}{\sqrt{n}},$$

where the value of z is determined by the confidence level required.

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PART 1 (Questions 1 to 8)

(66 marks)

Question 1 (7 marks)

Find $\frac{dy}{dx}$ for each of the following. There is no need to simplify your answers.

(a) $y = e^{x^2}$.

(2 marks)

(b) $y = 2x \sin 3x$.

(2 marks)

(c) $y = \frac{4x}{(6-x)^2}$.

(3 marks)

Question 2 (7 marks)

The table below gives the probability distribution function for a discrete random variable X. The values a and b represent two missing probabilities in the distribution function.

x	-1	0	2
$P(X = x)$	a	b	0.1

(a) Explain why $a + b = 0.9$.

(1 mark)

(b) Given that $\mu_X = 0$, write down a second equation for a and b.

(1 mark)

(c) Using the equation given in part (a) and your equation from part (b), find the values a and b.

(2 marks)

(d) Calculate σ_X.

(3 marks)

Question 3 (6 marks)

(a) Write the expression $5\ln 2+\frac{1}{2}\ln 16-\ln 8$ in the form $\ln k$.

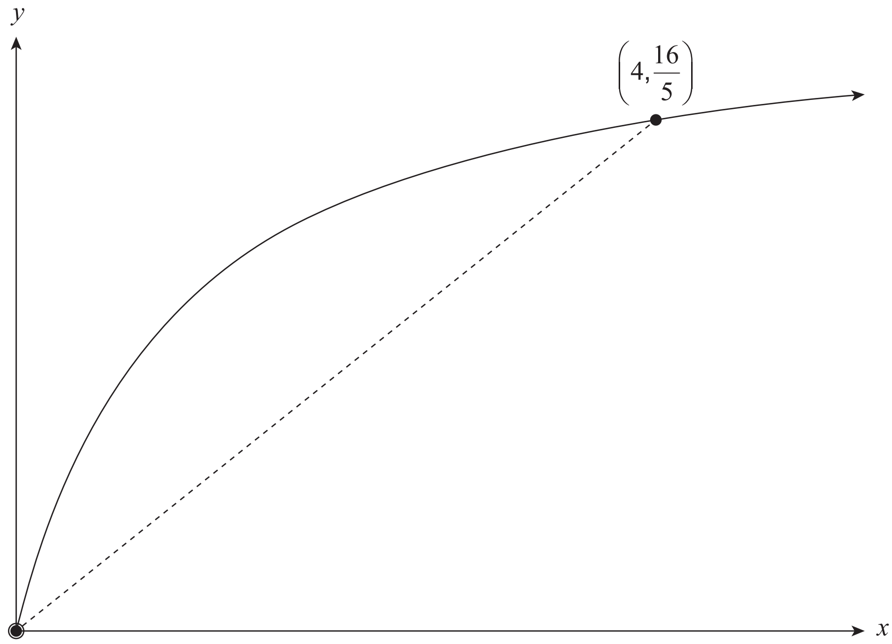
(3 marks)

(b) Solve the equation $\ln 16+2\ln x=0$ for $x>0$.

(3 marks)

Question 4 (8 marks)

The graph of $y = f(x)$, where $f(x) = \frac{4x}{x+1}$ for $x \geq 0$, is shown below.



The slope of the chord shown on the graph represents the average rate of change of $f(x)$ for the interval from $x = 0$ to $x = 4$.

(a) Find the slope of the chord.

(1 mark)

(b) Interpret the value of $f'(4)$.

(1 mark)

(c) Consider the graph of $y = f(x)$ on page 8, and the chord shown. The slope of the chord approximates the value of $f'(4)$.

On the graph, draw a chord that has a slope that more closely approximates the value of $f'(4)$. (1 mark)

(d) Using first principles, find $f'(4)$ for $f(x) = \frac{4x}{x+1}$.

(5 marks)

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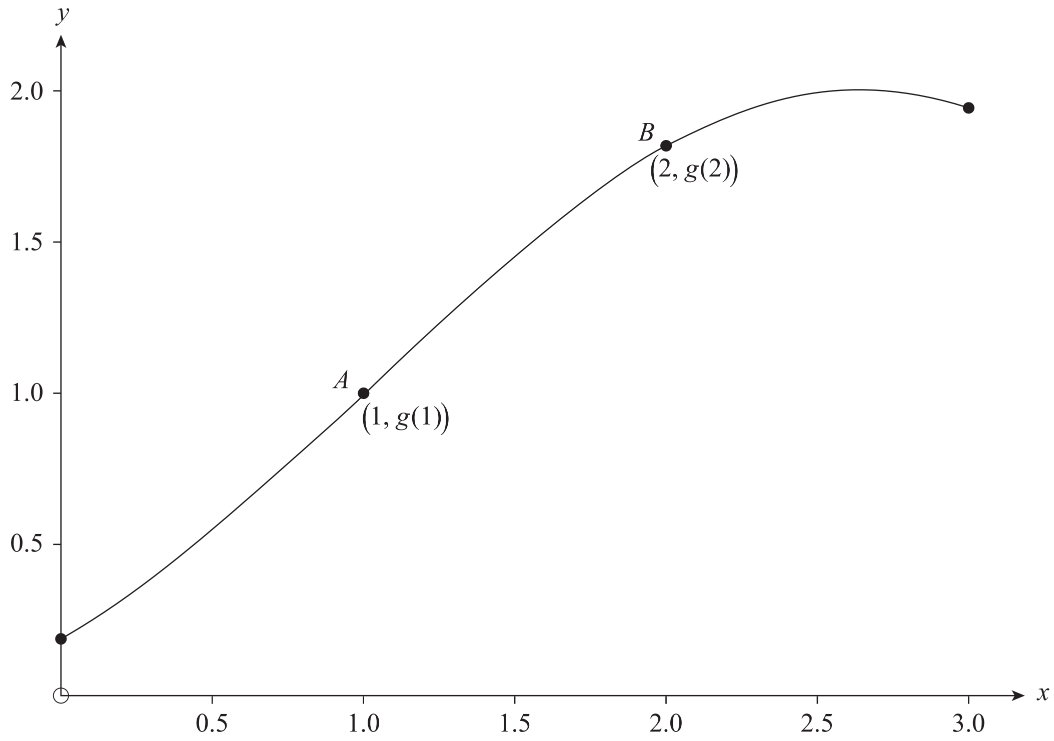
Question 5 (6 marks)

The graph of the curve $y = g(x)$ for $0 \le x \le 3$ is shown below.



Points A and B lie on the curve. Point A is the only point of inflection.

(a) Which *one* of the following statements is true? Tick the appropriate box, and justify your answer.

$g'(1) < g'(2)$ ☐

$g'(1) = g'(2)$ ☐

$g'(1) > g'(2)$ ☐

Justification:

(2 marks)

(b) Which *one* of the following statements is true? Tick the appropriate box, and justify your answer.

$g''(1) < 0$ ☐

$g''(1) = 0$ ☐

$g''(1) > 0$ ☐

Justification:

(2 marks)

(c) Which *one* of the following statements is true? Tick the appropriate box, and justify your answer.

$g''(2) < 0$ ☐

$g''(2) = 0$ ☐

$g''(2) > 0$ ☐

Justification:

(2 marks)

Question 6 (6 marks)

The owner of a local pet food shop is offering an incentive program to attract customers.

The shop owner has placed 50 balls in a bucket. Each ball is labelled with a percentage discount that the customer will be given after their purchase total is tallied. The customer draws a ball randomly from the bucket, and the percentage discount on the ball is applied to the purchase total. The ball is then placed back in the bucket before the next customer draws a ball.

Of the 50 balls in the bucket:

- one ball is labelled '100% discount'
- two balls are labelled '50% discount'
- four balls are labelled '25% discount'
- eight balls are labelled '20% discount'.

The remaining 35 balls are labelled '10% discount'.

(a) What is the probability that a customer will draw the ball labelled '100% discount'?

(1 mark)

(b) If 500 customers each draw one ball from the bucket, what is the probability that more than 10 of these customers will draw the ball labelled '100% discount'?

(2 marks)

(c) The shop owner could offer an alternative incentive program, in which a 15% discount is given to all customers.

In the long run, which of these two incentive schemes will cost the shop owner more? Explain your answer.

(3 marks)

Question 7 (14 marks)

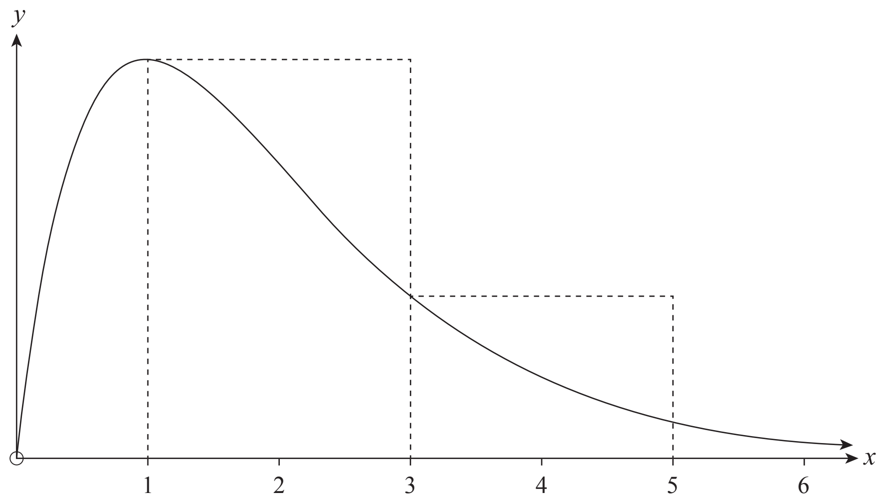
(a) (i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y = xe^{-x}$.

(2 marks)

(ii) Hence show that $\int xe^{-x}\mathrm{d}x = -xe^{-x} - e^{-x} + c$.

(3 marks)

The graph of $y = f(x)$, where $f(x) = xe^{-x}$, is shown below.



(b) An estimate is required for the area bounded by $f(x)$, the x-axis, and the vertical lines $x = 1$ and $x = 5$.

(i) Two rectangles, each 2 units wide, have been added to the graph to be used in the calculation of an overestimate for this area.

Calculate this overestimate by finding the sum (S) of the areas of these two rectangles, correct to three decimal places.

(2 marks)

(ii) A new overestimate of the same area can be calculated using *four* rectangles of equal width.

(1) On the graph above, draw four rectangles that could be used to produce a new overestimate. (1 mark)

(2) Calculate the new overestimate, giving your answer correct to three decimal places.

(2 marks)

(c) With reference to part (a)(ii), find the exact value of the area bounded by $f(x)$, the x-axis, and the vertical lines $x = 1$ and $x = 5$.

(3 marks)

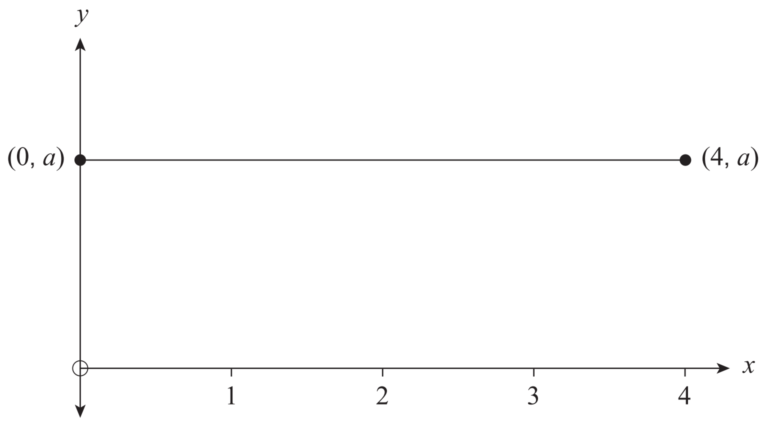
(d) Compare your overestimate calculations from part (b) with your answer to part (c).

Comment on the effect that increasing the number of rectangles used in your calculations has on the accuracy of the estimates obtained.

(1 mark)

Question 8 (12 marks)

(a) The graph below represents a continuous probability density function defined for $0 \le x \le 4$. Let this probability density function be $f(x) = a$.



(i) Find the value of a.

(2 marks)

(ii) (1) Write an integral expression for the mean (μ_X) of the distribution.

(1 mark)

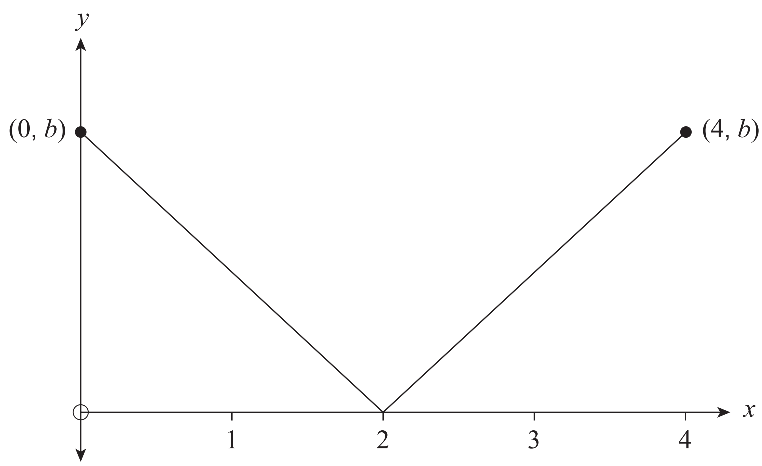
(2) Evaluate your expression to determine the μ_X of the distribution.

(2 marks)

(iii) Calculate the standard deviation (σ_X) of the distribution. Give your answer correct to one decimal place.

(2 marks)

(b) The graph below represents a different continuous probability density function defined for $0 \le x \le 4$. Let this probability density function be $g(x)$.



Show that the value of b is $\frac{1}{2}$.

(2 marks)

(c) (i) Using the information from parts (a) and (b), state which distribution you would expect to have the larger standard deviation.

(1 mark)

(ii) Justify your answer.

(2 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. Question 7(b)(ii)(2) continued).

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Mathematical Methods
2017

Question Booklet 2

2

- **Part 2** (Questions 9 to 14) 73 marks
- Answer ***all*** questions in Part 2
- Write your answers in this question booklet
- You may write on page 18 if you need more space
- Allow approximately 90 minutes
- Approved calculators may be used — complete the box below

Copy the information from your SACE label here

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   Model ____________

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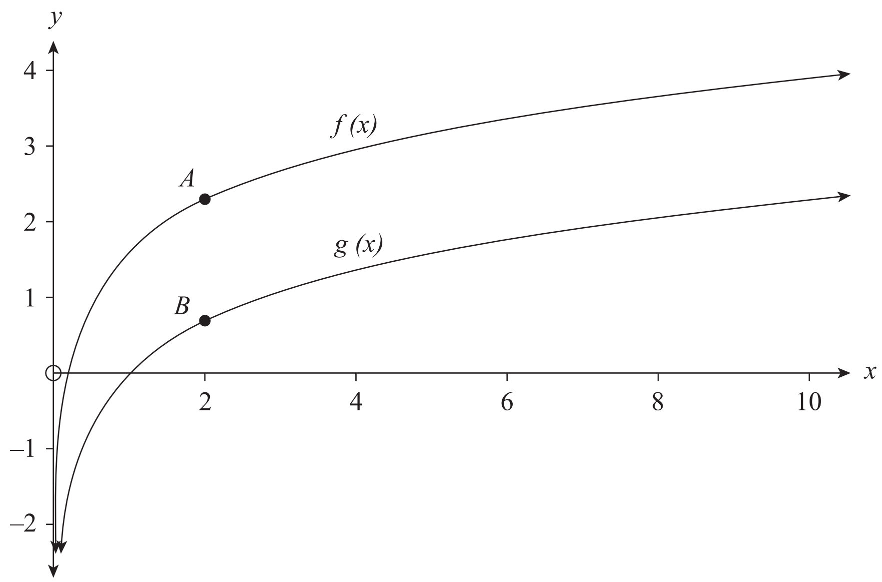
PART 2 (Questions 9 to 14)

(73 marks)

Question 9 (7 marks)

Consider the graphs of $f(x) = \ln 5x$ and $g(x) = \ln x$, as shown below.



There is a point labelled A at $(2, f(2))$ and a point labelled B at $(2, g(2))$.

(a) What is the exact distance between points A and B?

(2 marks)

(b) (i) Find the value of x such that $g'(x) = 2$.

(2 marks)

(ii) Let $h(x) = \ln kx$, where $k > 0$.

Find x such that $h'(x) = 2$.

(2 marks)

(c) Comment on the relationship between the graphs of $g(x) = \ln x$ and $h(x) = \ln kx$.

(1 mark)

Question 10 (10 marks)

The quality of sheep's wool is primarily based on the average diameter of its fibres, measured in microns (micrometres).

A wool producer wishes to make a claim about the average diameter of fibres in one bale of wool (Figure 1) sourced from one flock of Australian Merino sheep (Figure 2). Historically, the wool sourced from this flock has a fibre diameter that is normally distributed with a standard deviation of 2.5 microns.

The wool producer chooses 80 fibres randomly from the bale, then measures the diameter of each of these fibres and calculates their sample mean as $\bar{x}_{80} = 17.4$ microns.

Figure 1

Source: Adapted from © Patricia Hofmeester | Dreamstime.com

Figure 2

Source: © Steve Lovegrove | Dreamstime.com

(a) Based on this sample, calculate the 99% confidence interval for the population mean (μ).

(2 marks)

(b) Based on this sample, the 95% confidence interval for μ is $16.9 \leq \mu \leq 17.9$.

Explain why, when based on the same sample, a 95% confidence interval is wider than a 90% confidence interval.

(2 marks)

(c) Based on this sample, the wool producer labels this bale of wool as having an average fibre diameter of 18 microns or less.

A manufacturer can make a claim with '$k\%$ level of confidence' if it is supported by a $k\%$ confidence interval.

With reference to your answer to part (a) and the information provided in part (b), what can you conclude about the confidence with which the wool producer can label this bale of wool as having an average fibre diameter of 18 microns or less? Justify your answer.

(3 marks)

(d) What is the maximum level of confidence with which the wool producer can label this bale of wool as having an average fibre diameter of 17.5 microns or less?

(3 marks)

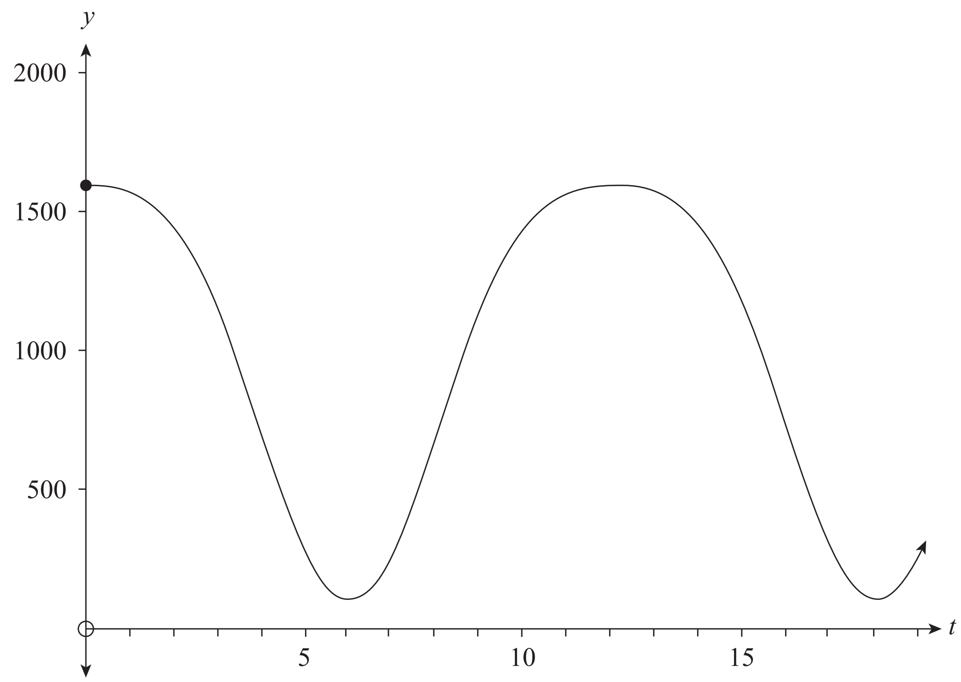
Question 11 (13 marks)

A major city has a wet season and a dry season. A manufacturer in this city produces umbrellas and sells these umbrellas throughout the year. The rate of sales of these umbrellas is not constant. Based on historical sales data, the rate at which the umbrellas are sold can be modelled by the function

$$f(t)=750\cos\left(\frac{\pi}{6}t\right)+150\cos\left(\frac{\pi}{3}(t-3)\right)+1000,$$

where t represents the time, in months, since 1 January 2016.

The graph of $y=f(t)$ is shown below.



(a) For what value of t is the rate of sales of umbrellas first at a minimum?

(1 mark)

(b) (i) Calculate how many umbrellas were sold during the first 3 months of 2016.

(2 marks)

(ii) On the graph above, represent the quantity that you calculated in part (b)(i). (1 mark)

The manufacturer produces 1000 umbrellas per month.

(c) Represent this information on the graph of $y = f(t)$ on page 6. (1 mark)

When the manufacturer produces more umbrellas than they sell, the surplus umbrellas are stored in a warehouse. When the manufacturer produces fewer umbrellas than they sell, the umbrellas that are stored in the warehouse are sold.

(d) When $t = 10$, which *one* of the following is happening? Tick the appropriate box.

The number of umbrellas stored in the warehouse is *increasing*. ☐

The number of umbrellas stored in the warehouse is *decreasing*. ☐ (1 mark)

(e) Show that the manufacturer sold the same number of umbrellas as they produced over the 12 months of 2016.

(2 marks)

(f) During 2016, for what values of t were the number of umbrellas stored in the warehouse increasing?

(2 marks)

(g) At $t = 0$, the warehouse contained 5250 umbrellas.

During 2016, what was the greatest number of umbrellas stored in the warehouse at one time?

(3 marks)

Question 12 (16 marks)

The distributor of a particular seed claims that, on average, 60% of the seeds are viable, meaning that they will grow into seedlings.

(a) Let X be the random variable representing the number of viable seeds per planting of eight seeds.

(i) State the distribution of X.

(1 mark)

(ii) Out of repeated plantings of eight seeds at a time, what is the expected number of viable seeds?

(1 mark)

(iii) The table below shows the probability of $k = 0, 1, 2, 3 \ldots$ seeds being viable.

Complete the table for $k = 4, 5, 6$.

k	$P(X = k)$
0	0.0007
1	0.0079
2	0.0413
3	0.1239
4	
5	
6	
7	0.0896
8	0.0168

(2 marks)

(b) A gardener plants eight of these seeds in his garden.

(i) What is the most likely number of these seeds that will be viable?

(1 mark)

(ii) What is the probability that six or fewer of these seeds will be viable?

(1 mark)

(c) A farmer bought a large bag of these seeds. She planted 100 of the seeds from this bag and observed that 53 of these seeds were viable.

(i) Calculate the proportion of planted seeds that were viable.

(1 mark)

(ii) Calculate a 95% confidence interval for the proportion of seeds in the bag that are viable.

(2 marks)

(iii) Based on the confidence interval that you calculated in part (c)(ii), can it be concluded that the distributor's claim that 'on average, 60% of the seeds are viable' is false? Explain your answer.

(2 marks)

(iv) Given an estimate p^*, show that the sample size n required to obtain a 95% confidence interval of width w for the population proportion is

$$n = \left(\frac{2 \times 1.96}{w}\right)^2 p^*(1 - p^*).$$

(3 marks)

(v) The farmer wishes to reduce the uncertainty in the estimate of the proportion of seeds in her bag that are viable.

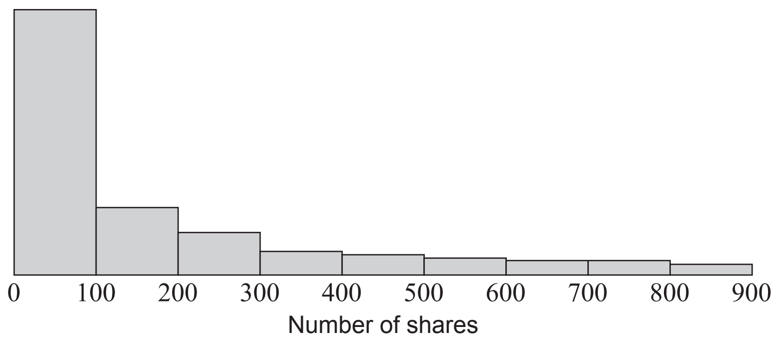
Using the information provided in part (c)(iv), or otherwise, calculate the smallest number of seeds that she would need to plant in order to have a confidence interval with width 0.1.

(2 marks)

Question 13 (11 marks)

Some companies advertise their products by creating online texts or humorous advertisements called ‘memes’ about a product. They post these memes on social media such as Weibo, Facebook, and Instagram in the hope that these will be ‘shared’ by a large number of social media users, thus promoting the product.

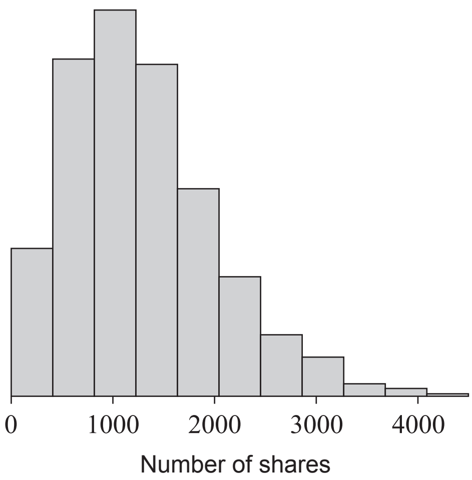
Scientists studied the number of ‘shares’ of memes for a particular product, called product C, and found that they follow the distribution depicted in the graph below. The distribution has a mean of 131.25 shares and a standard deviation of 221 shares.



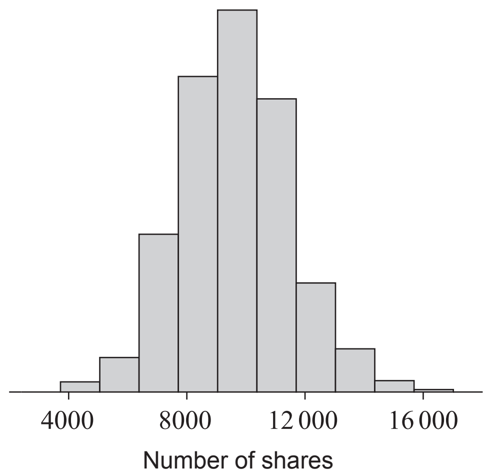
In an advertising campaign, companies may post a batch of memes on social media, in the hope that some of these will be shared by a large number of social media users. Let S_n represent the sum of the shares of a random sample of n product C memes.

(a) The following two histograms show the distributions of S_{10} and S_{75}:

Histogram A



Histogram B



Which histogram (**A** or **B**) corresponds to the distribution of S_{75}? Explain your answer.

(2 marks)

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(b) Show that the distribution of S_{75} has a mean of 9844 shares and a standard deviation of 1914 shares.

(2 marks)

(c) Consider the distribution of S_n.

(i) As n becomes sufficiently large, what will the distribution of S_n tend towards?

(1 mark)

(ii) Is $n = 75$ sufficiently large in this case? Explain your answer.

(2 marks)

An advertising campaign is considered to be successful when the combined total number of shares of the memes in the batch is $10\,000$ or greater.

It can be assumed that the memes in an advertising campaign can be considered a random sample drawn from the population of all product C memes.

(d) What is the probability that an advertising campaign consisting of a batch of 75 product C memes will be successful?

(2 marks)

(e) A company would like their advertising campaign to have a probability of success that is greater than 0.8.

Show that this would be achieved by placing a batch of at least 90 product C memes on social media.

(2 marks)

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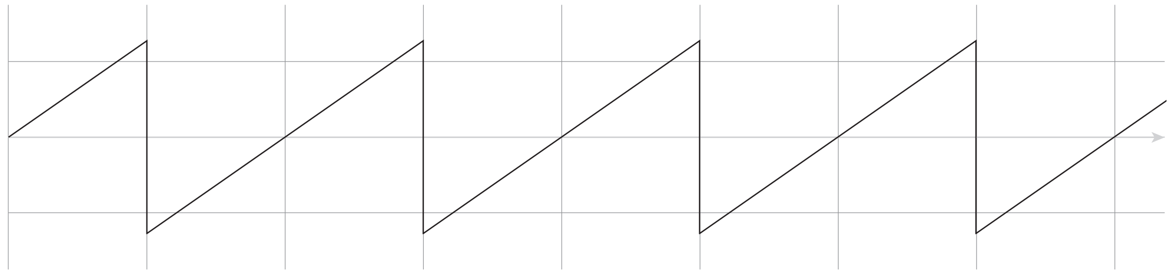
Question 14 (16 marks)

Jean-Baptiste-Joseph Fourier (1768–1830) was a French mathematician who explored how non-trigonometric functions can be modelled by the sum of trigonometric functions.

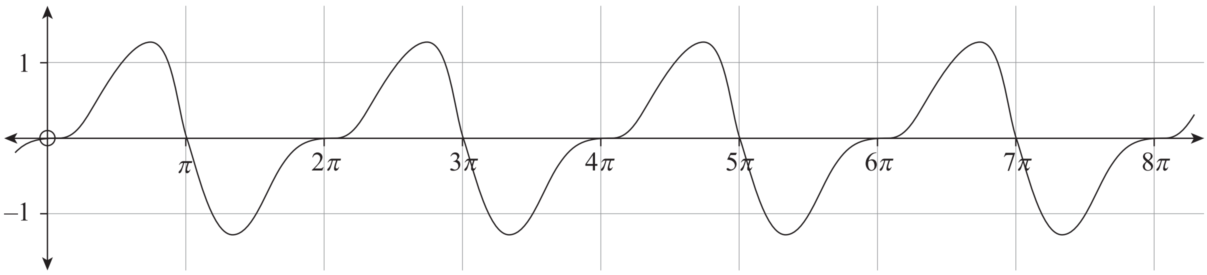
One function that can be modelled by a sum of trigonometric functions is a sawtooth wave, which is used in music synthesisers.

One example of a sawtooth wave is shown below.

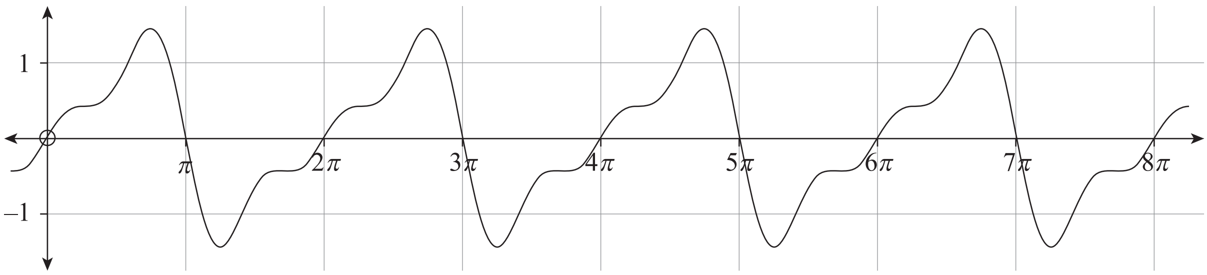


Consider the following sequence of graphs, which becomes progressively more triangular — and hence appears more like a sawtooth wave — as additional terms are added. Each function has an associated k value, where k represents the number of terms in the function.

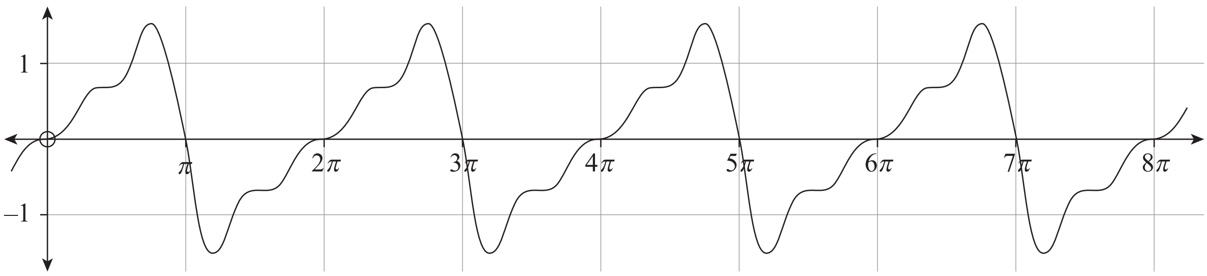
$f(x) = \sin x - \frac{\sin 2x}{2}, \; k = 2$



$g(x) = \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3}, \; k = 3$



$h(x) = \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4}, \; k = 4$



(a) (i) Find $f'(x)$.

(1 mark)

(ii) Hence show that $f'(\pi) = -2$.

(1 mark)

(iii) On the graph of $f(x)$ on page 14, sketch the tangent to $f(x)$, where $x = \pi$. (1 mark)

(b) (i) Find $g'(x)$.

(1 mark)

(ii) Find $h'(x)$.

(1 mark)

(iii) Hence complete the following table:

$f'(\pi)$	$g'(\pi)$	$h'(\pi)$
-2		

(2 marks)

(iv) Explain the effect of increasing the number of terms (i.e. increasing k from 2 to 3 to 4) on the slope of the tangent to each graph at $x = \pi$.

(1 mark)

(c) Consider the following function associated with $k=7$:

$$p(x)=\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\frac{\sin 5x}{5}-\frac{\sin 6x}{6}+\frac{\sin 7x}{7}.$$

Make a conjecture about the value of $p'(\pi)$.

(1 mark)

(d) Let $r(x)$ be a function of this type, with k terms, where k is any positive integer. The general form of $r(x)$ is therefore:

$$r(x)=\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\frac{\sin 5x}{5}-\frac{\sin 6x}{6}+\ldots+(-1)^{k-1}\frac{\sin kx}{k}.$$

(i) Make a conjecture about the value of the derivative $r'(\pi)$ for any value of k.

(1 mark)

(ii) Let y be the k^{th}, or final, term of $r(x)$ above, where k is a constant:

$$y=(-1)^{k-1}\frac{\sin kx}{k}.$$

Show that when $x=\pi$, $\frac{dy}{dx}=-1$ for all values of k.

(3 marks)

(iii) Hence justify your conjecture in part (d)(i) about the value of $r'(\pi)$ for any value of k.

(1 mark)

(iv) As the number of terms, k, becomes very large:

(1) describe what happens to the tangent to $r(x)$ at $x = \pi$.

(1 mark)

(2) identify the feature of the sawtooth wave that is represented by the tangent to $r(x)$ at $x = \pi$.

(1 mark)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. Question 12(c)(iv) continued).