South Australian
Certificate of Education

1

Mathematical Methods
2018

Question booklet 1

- **Part 1** (Questions 1 to 9) 64 marks
- Answer ***all*** questions in Part 1
- Write your answers in this question booklet
- You may write on pages 12 and 22 if you need more space
- Allow approximately 90 minutes
- Approved calculators may be used — complete the box below

Examination information

Materials
- Question booklet 1 (Part 1)
- Question booklet 2 (Part 2)
- SACE registration number label

Reading time
- 10 minutes
- You may begin writing during this time
- You may begin using an approved calculator during this time

Writing time
- 3 hours
- Show appropriate working and steps of logic in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams

Total marks 142

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Attach your SACE registration number label here

Graphics calculator

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   Model ____________

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   Model ____________

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Remove this page from the booklet by tearing along the perforations.

LIST OF MATHEMATICAL FORMULAE FOR USE IN STAGE 2 MATHEMATICAL METHODS

Properties of derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\,g(x)\right)=f'(x)\,g(x)+f(x)\,g'(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)\,g(x)-f(x)\,g'(x)}{\left[g(x)\right]^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}f\left(g(x)\right)=f'\left(g(x)\right)g'(x)$$

Quadratic equations

If $ax^2+bx+c=0$ then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Discrete random variables

The mean or expected value of a discrete random variable is:

$$\mu_X=\sum x.p(x),$$

where $p(x)$ is the probability function for achieving result x.

The standard deviation of a discrete random variable is:

$$\sigma_X=\sqrt{\sum\left[x-\mu_X\right]^2 p(x)},$$

where μ_X is the expected value and $p(x)$ is the probability function for achieving result x.

Bernoulli distribution

The mean of the Bernoulli distribution is p, and the standard deviation is:

$$\sqrt{p(1-p)}.$$

Binomial distribution

The mean of the binomial distribution is np, and the standard deviation is:

$$\sqrt{np(1-p)},$$

where p is the probability of success in a single Bernoulli trial and n is the number of trials.

The probability of k successes from n trials is:

$$\Pr(X=k)=C_k^n\,p^k(1-p)^{n-k},$$

where p is the probability of success in the single Bernoulli trial.

Population proportions

The sample proportion is $\hat{p}=\dfrac{X}{n}$,

where sample of size n is chosen, and X is the number of elements with a given characteristic.

The distribution of a sample proportion has a mean of p and a standard deviation of

$$\sqrt{\frac{p(1-p)}{n}}.$$

The upper and lower limits of a confidence interval for the population proportion are:

$$\hat{p}-z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\le p\le\hat{p}+z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},$$

where the value of z is determined by the confidence level required.

Continuous random variables

The mean or expected value of a continuous random variable is:

$$\mu_X=\int_{-\infty}^{\infty}x\,f(x)\,\mathrm{d}x,$$

where $f(x)$ is the probability density function.

The standard deviation of a continuous random variable is:

$$\sigma_X=\sqrt{\int_{-\infty}^{\infty}\left[x-\mu_X\right]^2 f(x)\,\mathrm{d}x},$$

where $f(x)$ is the probability density function.

Normal distributions

The probability density function for the normal distribution with the mean μ and the standard deviation σ is:

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

All normal distributions can be transformed to the standard normal distribution with $\mu=0$ and $\sigma=1$ by:

$$Z=\frac{X-\mu}{\sigma}.$$

Sampling and confidence intervals

If $\bar{x}$ is the sample mean and s the standard deviation of a suitably large sample, then the upper and lower limits of the confidence interval for the population mean are:

$$\bar{x}-z\frac{s}{\sqrt{n}}\le\mu\le\bar{x}+z\frac{s}{\sqrt{n}},$$

where the value of z is determined by the confidence level required.

Question 1 (8 marks)

For the functions below, determine $\frac{\mathrm{d}y}{\mathrm{d}x}$. You do not need to simplify your answers.

(a) $y = 5 + 4\sqrt{x} - \frac{10}{x^3}$.

(2 marks)

(b) $y = \frac{e^{0.5x}}{0.7x + e^x}$.

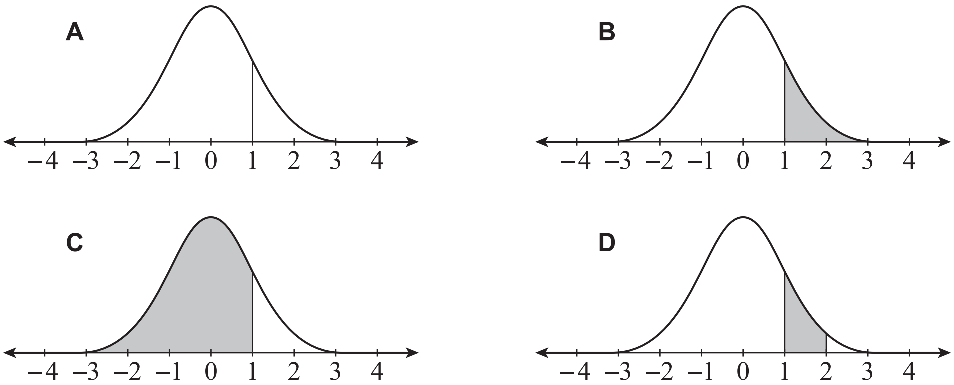
(3 marks)

(c) $y = \ln\left(x^2\sqrt{1-4x}\right)$.

(3 marks)

Question 2 (5 marks)

(a) Each of the graphs below represents the normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$.

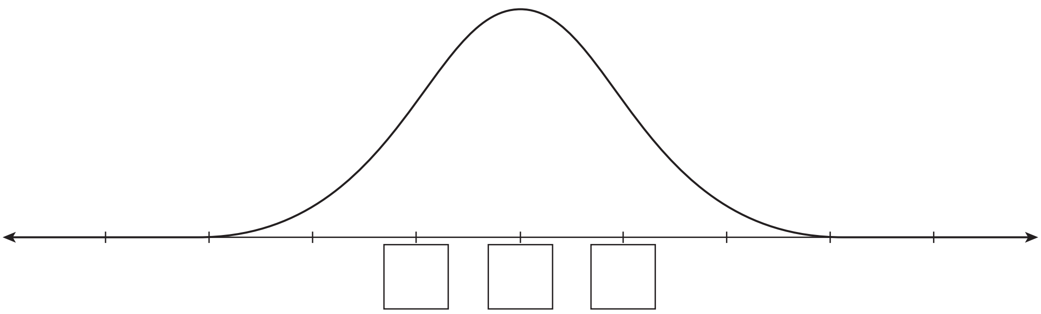


Z is a normally distributed random variable with mean $\mu = 0$ and standard deviation $\sigma = 1$.

Which *one* of graphs **A**, **B**, **C**, and **D** best illustrates $\Pr(Z \geq 1)$?

(1 mark)

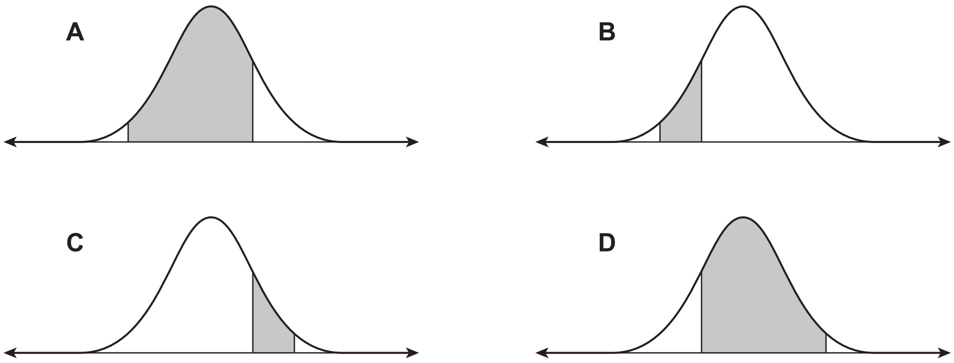
(b) The graph below represents the normal distribution with mean $\mu = 40$ and standard deviation $\sigma = 10$.



On the graph above, write a number in each box to provide a horizontal scale for this distribution.

(2 marks)

(c) Each of the graphs below represents the normal distribution with mean $\mu = 12$ and standard deviation $\sigma = 2$.

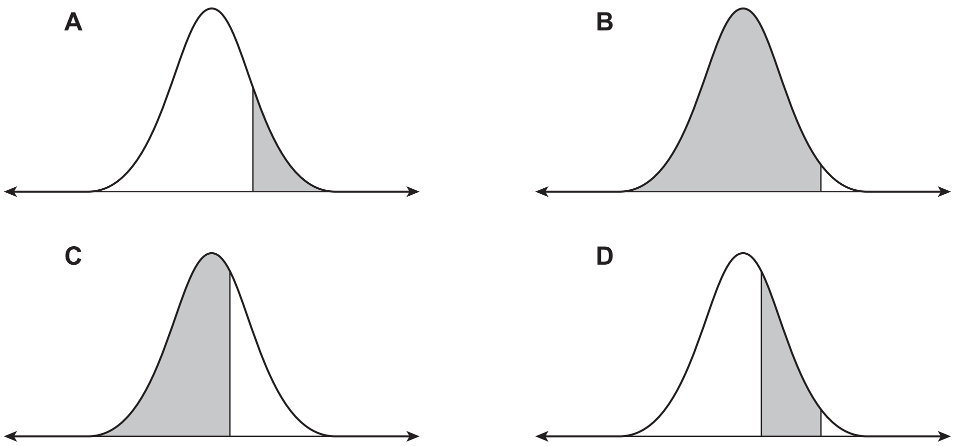


Y is a normally distributed random variable with mean $\mu = 12$ and standard deviation $\sigma = 2$.

Which *one* of graphs **A**, **B**, **C**, and **D** above best illustrates $\Pr(10 \le Y \le 16)$?

(1 mark)

(d) Each of the graphs below represents the normal distribution with mean $\mu = 120$ and standard deviation $\sigma = 40$.



X is a normally distributed random variable with mean $\mu = 120$ and standard deviation $\sigma = 40$.

Which *one* of graphs **A**, **B**, **C**, and **D** best illustrates $\Pr(X \le 140)$?

(1 mark)

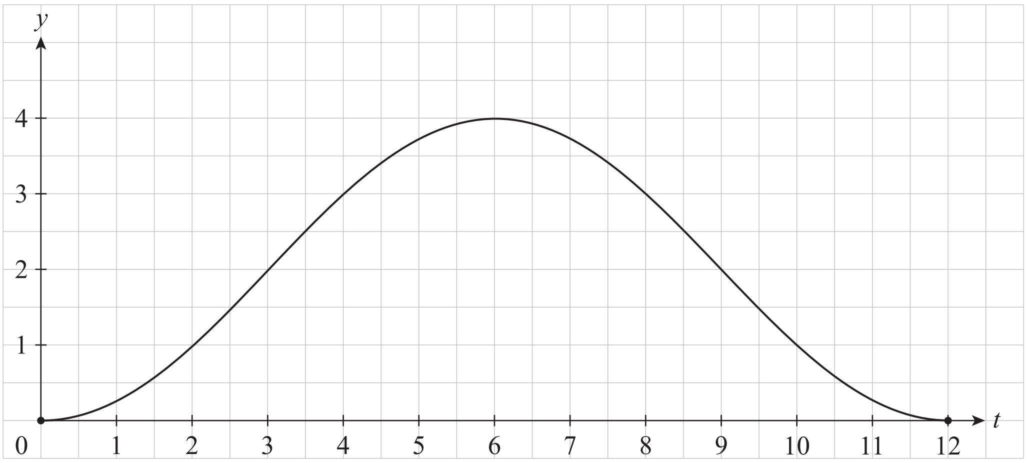
Question 3 (8 marks)

A stormwater detention tank is used to regulate the flow of stormwater into a drainage system. The tank fills during periods of heavy rainfall and empties when the rainfall becomes less heavy.

The volume of stormwater, V (in kilolitres), contained in a detention tank t hours after the start of a particular period of heavy rainfall can be modelled by the function

$$V(t) = 2\sin\left(\frac{\pi}{6}(t-3)\right) + 2, \text{ where } 0 \le t \le 12.$$

The graph of $y = V(t)$ is shown below.



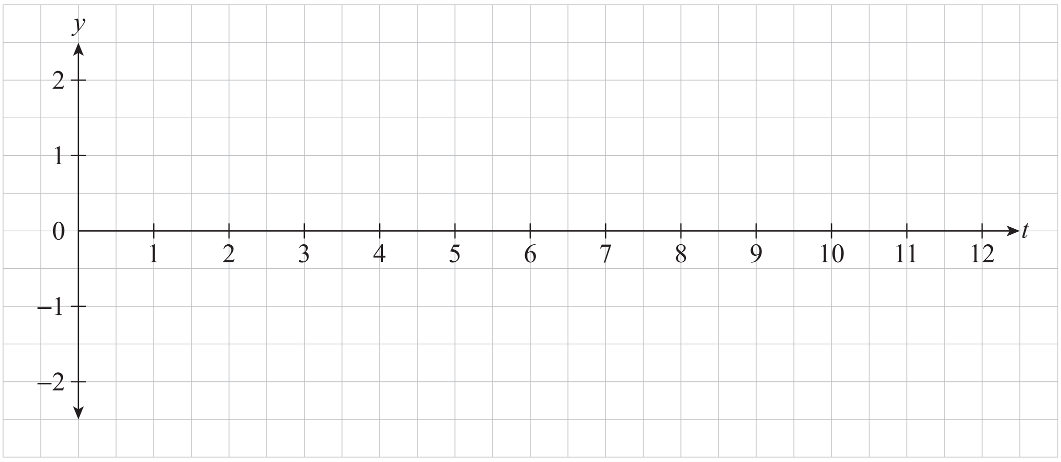
(a) (i) Write down an expression for $\frac{dV}{dt}$.

(2 marks)

(ii) What does $\frac{dV}{dt}$ represent in this context?

(1 mark)

(b) On the axes below, sketch the graph of $y = \frac{\mathrm{d}V}{\mathrm{d}t}$ for $0 \le t \le 12$.



(2 marks)

(c) (i) This period of heavy rainfall started at 11 am.

At what time of day was the rate of flow of stormwater into the detention tank greatest?

(2 marks)

(ii) Which feature of the graph of $y = V(t)$ on page 8 is associated with your answer to part (c)(i)?

(1 mark)

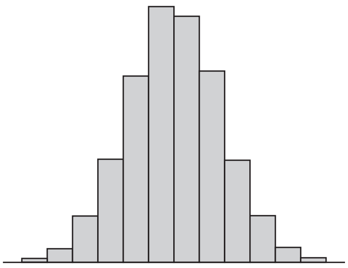
Question 4 (8 marks)

A manufacturer produces snack-sized packets of nuts. The net weight, W, of a randomly selected packet of nuts has a distribution with mean $\mu = 18.2$ g and standard deviation $\sigma = 0.3$ g.

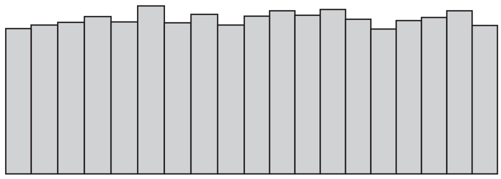
The manufacturer monitors the net weight of the packets of nuts by taking random samples of 15 packets of nuts and recording the mean net weight, $\overline{W}_{15}$, of each sample.

(a) The two histograms below represent the distributions of W and $\overline{W}_{15}$.

Histogram A



Histogram B



Which histogram (**A** or **B**) represents the distribution of $\overline{W}_{15}$?

(1 mark)

Let $\overline{W}_{25}$ represent the mean net weight of random samples of 25 packets of nuts.

(b) Calculate the mean and standard deviation of $\overline{W}_{25}$.

(2 marks)

(c) Assuming that $\overline{W}_{25}$ is normally distributed, determine $\Pr\left(\overline{W}_{25} \leq 18\right)$.

(1 mark)

(d) The manufacturer sells these packets of nuts in boxes. Each box contains 25 randomly selected packets of nuts. The label on each box states: 'Total net weight: at least 450 g'.

With reference to your answer to part (c), comment on whether or not this statement is reasonable.

(2 marks)

(e) With reference to the histograms provided in part (a), explain why it is reasonable to assume that $\overline{W}_{25}$ is normally distributed.

(2 marks)

You may write on this page if you need more space to finish your answers to any of the questions in Part 1. Make sure to label each answer carefully (e.g. 3(a)(i) continued).

Question 5 (5 marks)

Using first principles, find $f'(x)$ for $f(x)=\dfrac{5}{x^2}$.

(5 marks)

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Question 6 (7 marks)

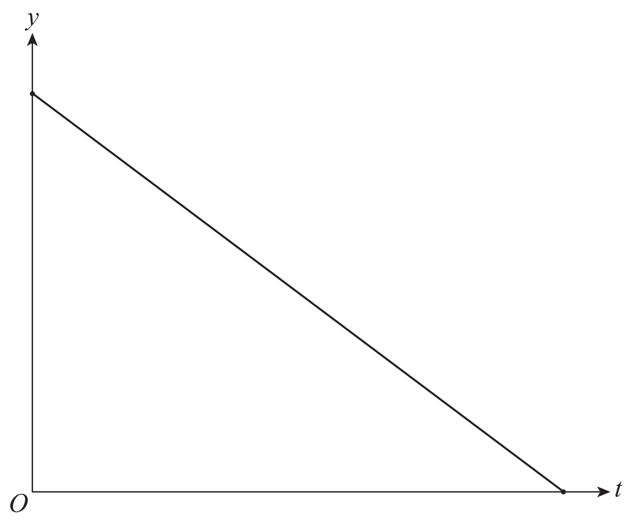
Under normal road conditions, the velocity of a motor vehicle, once its brakes 'lock', decreases at a constant rate of $6\ \text{m}\,\text{s}^{-2}$ until the vehicle stops.

The velocity of a vehicle t seconds after its brakes become locked can be modelled by the function

$$v(t) = -6t + k\ \text{m}\,\text{s}^{-1},$$

where k is a real number representing the initial velocity of the vehicle.

The graph of $y = v(t)$ is shown below.



The graph of $y = v(t)$ has a horizontal axis intercept.

(a) Find the coordinates of this intercept, in terms of k.

(2 marks)

(b) (i) Write down an integral expression for d, the distance travelled by a vehicle from $t = 0$ until it stops.

(1 mark)

(ii) Complete this integration to find d, in terms of k.

(2 marks)

(c) A particular vehicle is travelling under normal road conditions. Once its brakes become locked, the vehicle travels 20 metres before stopping.

Using your answer to part (b)(ii), find k. Give your answer correct to one decimal place.

(2 marks)

Question 7 (8 marks)

(a) Determine $g'(x)$, given that $g(x) = \dfrac{(\ln x)^2}{2}$.

(2 marks)

(b) The value of $A(k)$ is given by

$$A(k) = \int_1^e \frac{\ln x + k}{x}\,dx,$$

where k is a positive integer.

(i) Complete the table below by evaluating $A(k)$ for $k = 2$ and $k = 3$.

k	1	2	3
$A(k)$	1.5		

(1 mark)

(ii) Hence make a conjecture about $A(k)$ for any value of k.

(1 mark)

(iii) Prove or disprove your conjecture from part (b)(ii).

(4 marks)

Question 8 (8 marks)

Based on historical breed data, the weight of adult male Persian cats is distributed normally with mean $\mu = 6.0$ kg and standard deviation $\sigma = 0.55$ kg. Adult male Persian cats that weigh more than 7 kg are considered to be overweight.

(a) Based on this distribution, what is the proportion of adult male Persian cats that are overweight?

(2 marks)

In 2017, researchers undertook a study into the weights of adult male Persian cats. In this study, 70 adult male Persian cats were randomly selected and weighed. This sample of 70 cats had a mean weight of $\bar{x} = 6.7$ kg.

(b) Assuming that the population standard deviation is $\sigma = 0.55$ kg, use the sample data to calculate a 95% confidence interval for μ, the mean weight of adult male Persian cats in 2017.

(2 marks)

The researchers wrote a press release entitled ‘Increase in mean weight of adult male Persian cats’.

(c) Does the confidence interval that you calculated in part (b) support the title of this press release? Justify your answer.

(2 marks)

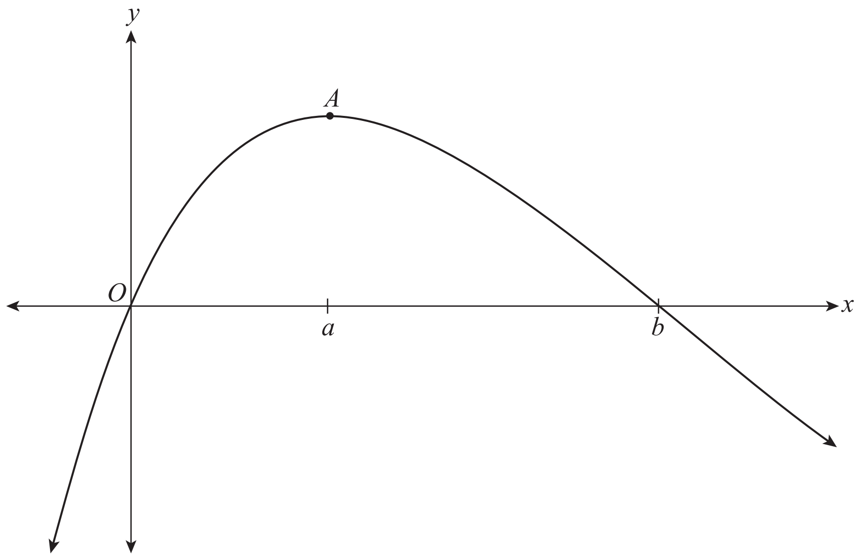
(d) The press release states that between 22% and 38% of adult male Persian cats are overweight.

Provide mathematical calculations that support this statement.

(2 marks)

Question 9 (7 marks)

Consider the function $f(x)$. The graph of its *derivative*, $y = f'(x)$, is shown below.
The graph intersects the x-axis at the origin (O) and at $x = b$. The point A, where $x = a$, is a local maximum of the graph of $y = f'(x)$.

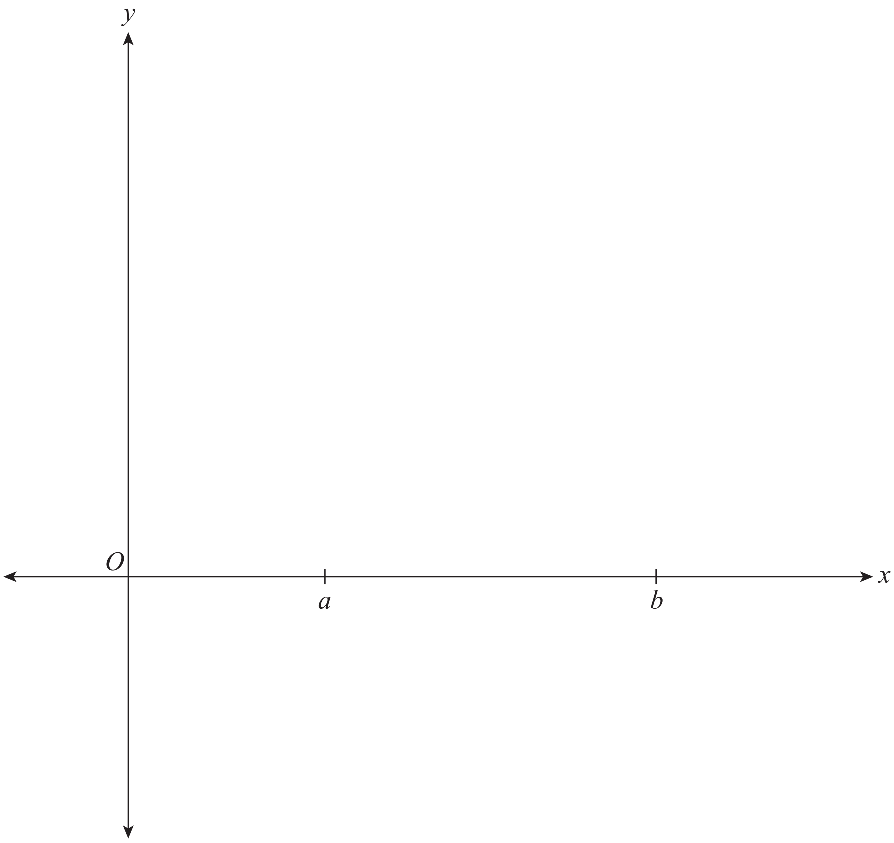


(a) Complete the table below by indicating whether $f'(x)$ and $f''(x)$ are positive (+), negative (–), or zero (0) when $x = a$ and when $x = b$.

x	a	b
$f'(x)$		
$f''(x)$		

(4 marks)

(b) On the axes below, sketch a possible graph of $y = f(x)$ that passes through the origin. Clearly show the shape of the graph in the vicinities of the origin, $x = a$, and $x = b$.



(3 marks)

You may write on this page if you need more space to finish your answers to any of the questions in Part 1. Make sure to label each answer carefully (e.g. 3(a)(i) continued).

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2018

Question booklet 2

- **Part 2** (Questions 10 to 16) 78 marks
- Answer ***all*** questions in Part 2
- Write your answers in this question booklet
- You may write on pages 16 and 20 if you need more space
- Allow approximately 90 minutes
- Approved calculators may be used — complete the box below

2

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Graphics calculator

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 Model ____________

Question 10 (13 marks)

Consider the function $g(x) = \sin x - x\cos x$.

(a) Show that $g'(x) = x\sin x$.

(2 marks)

Let X be a continuous random variable. X has the probability density function $f(x) = k\sin x$, where $0 \le x \le \pi$ and k is a real and positive constant.

(b) (i) Show that $k = \frac{1}{2}$.

(3 marks)

(ii) Calculate $\Pr\left(\frac{\pi}{4} \le X \le \frac{3\pi}{4}\right)$.

(2 marks)

(c) (i) Write down an integral expression for the mean, μ_X.

(1 mark)

(ii) Evaluate the integral expression that you wrote down in part (c)(i) to show that $\mu_X = \frac{\pi}{2}$.

(3 marks)

(d) Calculate the standard deviation, σ_X.

(2 marks)

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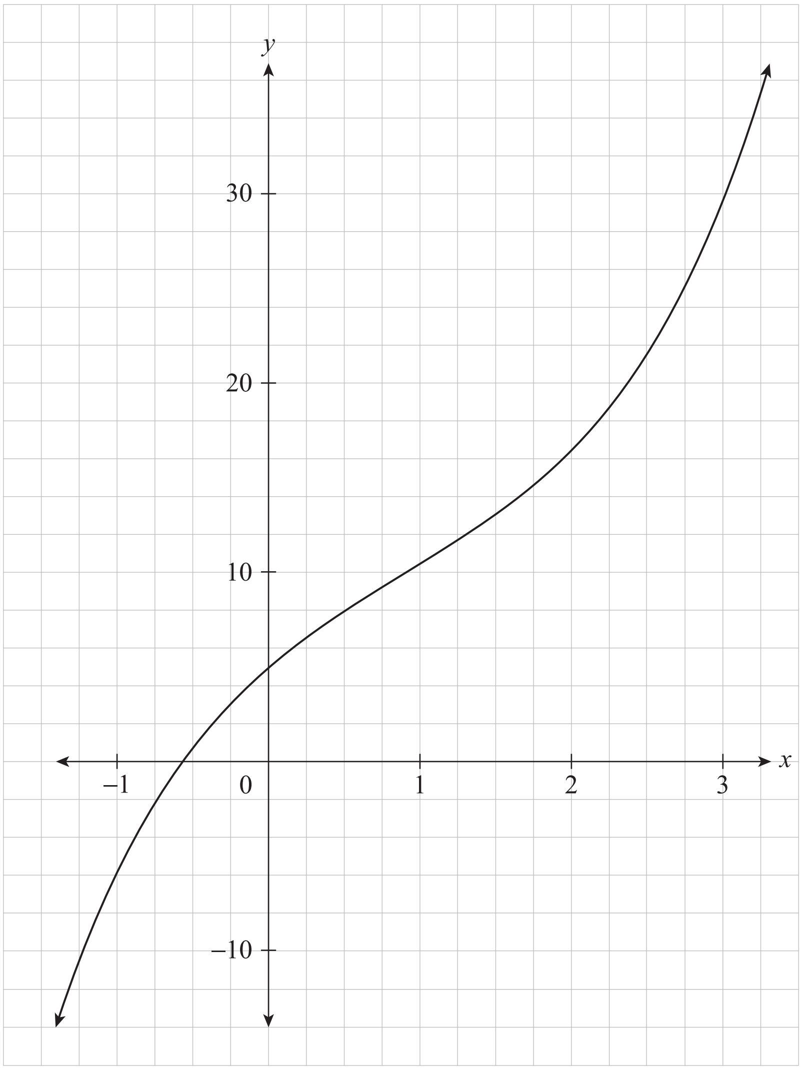
Question 11 (10 marks)

Consider the function $f(x) = e^x + 10 - 6e^{-x}$. The graph of $y = f(x)$ is shown below.



(a) An estimate is required for A, the area between the graph of $y = f(x)$ and the x-axis from $x = 1$ to $x = 3$.

(i) An overestimate of area A is to be calculated, using four rectangles of equal width.

On the graph above, draw the four rectangles used to obtain this overestimate. (1 mark)

(ii) Calculate this overestimate, correct to four significant figures.

(2 marks)

(b) (i) Determine $f''(x)$.

(2 marks)

(ii) The solution to $f''(k)=0$, correct to three significant figures, is $k=0.896$.

Determine the exact value of k.

(2 marks)

(iii) For what values of x is $f''(x)>0$?

(1 mark)

(c) An underestimate of area A could also be calculated, using four rectangles of equal width.

With reference to your answer to part (b)(iii), explain which estimate — this underestimate or the overestimate calculated in part (a)(ii) — would be closer to area A. Do ***not*** calculate this underestimate.

(2 marks)

Question 12 (6 marks)

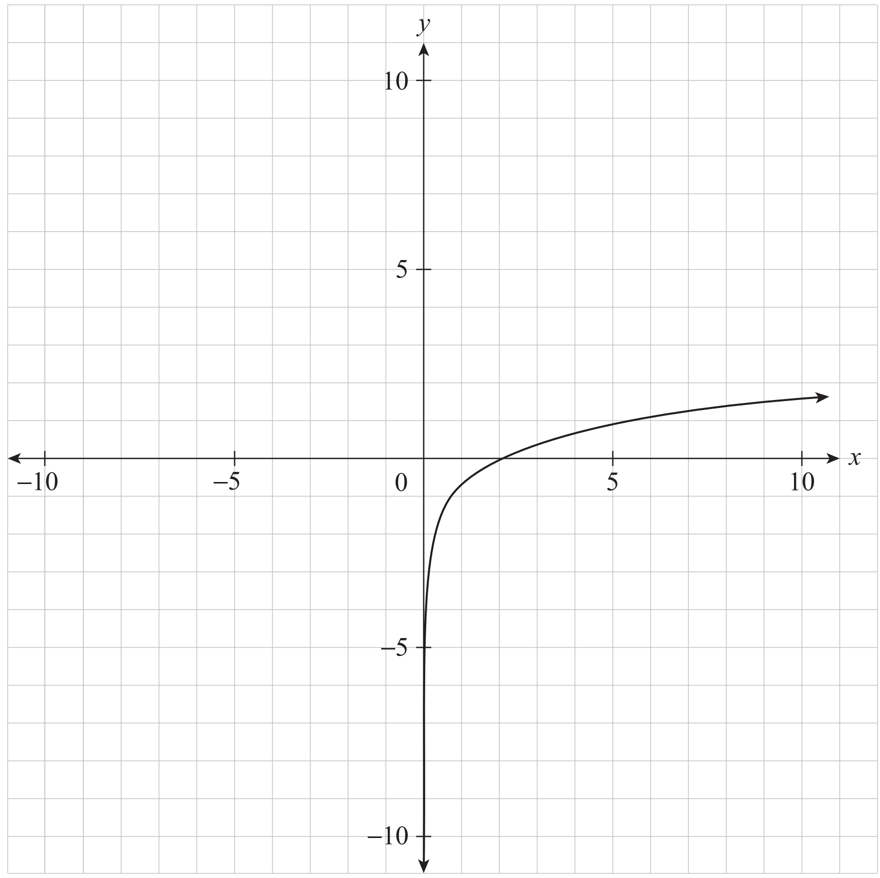
(a) Simplify the following expression:

$$\ln\left(\frac{1}{2}x\right)^2 + \ln\left(\frac{1}{2}x\right).$$

Write your answer in the form $a\ln bx$, where a and b are real numbers.

(1 mark)

(b) The graph of the function $y_1 = \ln\left(\frac{1}{2}x\right)$ is shown below.



(i) On the axes above, sketch the graph of the function $y_2 = \ln\left(\frac{1}{2}x\right)^2 + \ln\left(\frac{1}{2}x\right)$. (1 mark)

(ii) Describe the relationship between the graphs of the functions

$$y_1 = \ln\left(\frac{1}{2}x\right)$$

and

$$y_2 = \ln\left(\frac{1}{2}x\right)^2 + \ln\left(\frac{1}{2}x\right).$$

(2 marks)

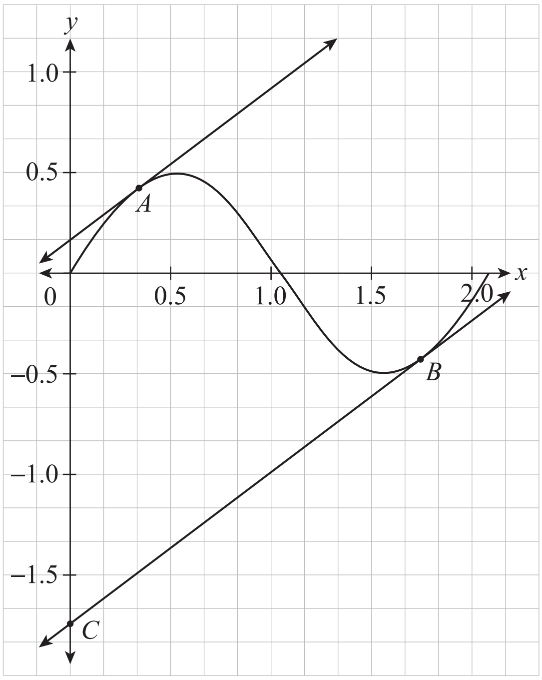
(c) Find the exact value of x such that

$$\ln\left(\frac{1}{2}x\right)^2 + \ln\left(\frac{1}{2}x\right) = 3.$$

(2 marks)

Question 13 (11 marks)

Consider the function $f(x)=\frac{1}{2}\sin(3x)$ for $0\le x\le\frac{2\pi}{3}$. The graph of $y=f(x)$ is shown below, along with parallel tangents to this graph at points A and B.



(a) (i) Find $f'(x)$.

(2 marks)

(ii) Let m represent the gradient of any tangent to the graph of $y=f(x)$. The value of m is bounded such that $p\le m\le q$.

Determine p and q.

(2 marks)

(b) For the parallel tangents to the graph of $y = f(x)$ that are shown on page 8, $m = \frac{3}{4}$.

Find the exact coordinates of point B.

(4 marks)

(c) The tangent to the graph of $y = f(x)$ at point B intersects the y-axis at point $C(0, c)$.

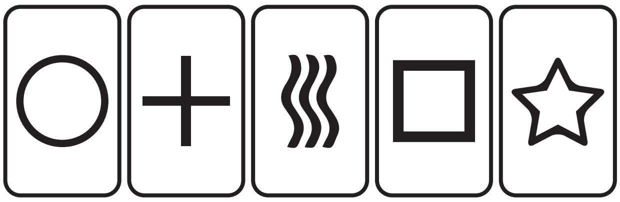
Find the exact value of c.

(3 marks)

Question 14 (12 marks)

Extrasensory perception (ESP) is defined as the reception of information without the use of the recognised physical senses such as sight or hearing.

One test for ESP uses a set of Zener cards. A set of 25 Zener cards contains five of each of the cards illustrated below. In this test, a test administrator randomly selects one card from the set of 25 cards. The card is concealed from the test subject, who attempts to correctly identify the symbol that is printed on the card. The test administrator records whether or not the test subject correctly identifies the symbol printed on the card.



Source: adapted from Ryazanov, M 2014, 'File: Zener cards', *Wikimedia Commons, the free media repository*, viewed 28 August 2018, https://commons.wikimedia.org/wiki/File:Zener_cards_(color).svg

(a) Explain why the test subject's attempt to correctly identify which of the five symbols is printed on a selected card constitutes a Bernoulli trial.

(1 mark)

To complete the test, the process described above is undertaken 25 times, with cards being returned to the set between each attempt.

Let X represent the number of correct identifications out of 25 attempts, where the test subject is identifying symbols at random with the probability of correct identification $p = 0.2$.

(b) What is the probability that the test subject makes:

(i) exactly five correct identifications?

(1 mark)

(ii) no more than seven correct identifications?

(2 marks)

(c) In the context of this test, write a probability statement that is equivalent to:

(i) $C_8^{25}(0.2)^8(0.8)^{17} = 0.0623.$

(1 mark)

(ii) $1 - C_0^{25}(0.2)^0(0.8)^{25} = 0.996.$

(1 mark)

To be successful in this test, the test subject must make at least k correct identifications, where k is chosen so that the chance of being successful by the random identification of symbols is less than 2.5%.

(d) Calculate k^*, the minimum value of k. Support your answer with probability calculations correct to four significant figures.

(2 marks)

Students at a high school watched a television program investigating ESP. One student, Ari, undertook an ESP test using Zener cards. She made k^* correct identifications — where k^* is the value calculated in part (d) — and claimed to possess ESP.

(e) Based on the information in part (d), what can be said about Ari's claim?

(1 mark)

(f) If 624 students at this high school undertook an ESP test using Zener cards and were identifying symbols at random, what is the probability that one or more of these students would make at least k^* correct identifications?

(2 marks)

The principal of the school suggests that Ari was identifying symbols at random and hence does not possess ESP.

(g) Explain how your answer to part (f) supports the principal's suggestion.

(1 mark)

Question 15 (14 marks)

'Wacky Quackers' is a popular amusement park game in Australia. In this game, a player selects from a very large number of identical-looking plastic ducks that are floating on a pool of water. Each duck has the number 1, 2, 5, or 10 printed on its base. The numbers cannot be seen by the player when making their selection.

The number that is printed on the selected duck will be used to determine the player's prize. These numbers are distributed according to the table below.

x	1	2	5	10
$\Pr(X = x)$	0.5	0.2	0.2	0.1

Source: adapted from © Iprintezis | Dreamstime.com

(a) (i) Calculate μ_X.

(2 marks)

(ii) Calculate σ_X.

(2 marks)

A player can pay $\$10$ and randomly select three ducks. This player wins a prize that has a dollar ($\$$) value equal to the sum of the numbers printed on the three ducks.

three ducks $10

(b) (i) What is the probability that a player who randomly selects three ducks will win a prize that has a value of $\$30$?

(2 marks)

(ii) What is the probability that a player who randomly selects three ducks will win a prize that has a value of at least $\$5$?

(3 marks)

Let the discrete random variable S_3 represent the sum of the numbers printed on three randomly selected ducks.

(c) (i) Calculate μ_{S_3}.

(1 mark)

(ii) Calculate σ_{S_3}.

(1 mark)

Alternatively, a player can pay $16 and randomly select five ducks. This player wins a prize that has a dollar value equal to the sum of the numbers printed on the five ducks.

five ducks $16

(d) Calculate the expected value of the sum of the numbers printed on the five randomly selected ducks.

(1 mark)

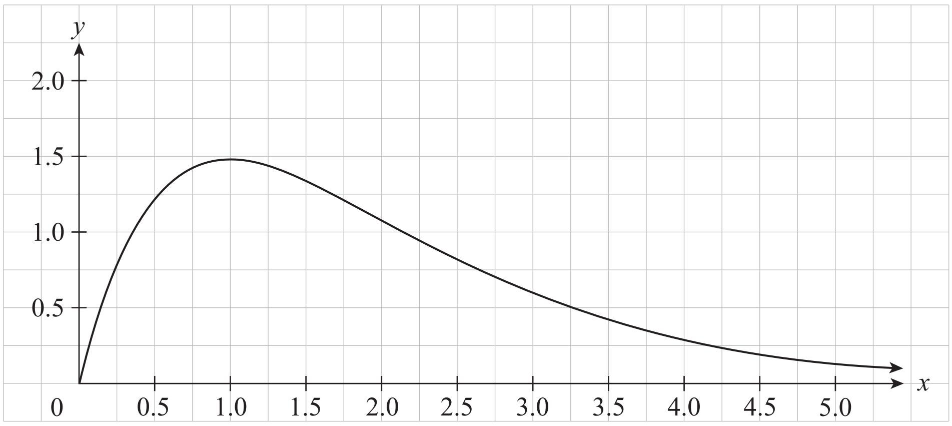
(e) Using your answers to part (c) and part (d), determine which *one* of the two options — three ducks for $10 or five ducks for $16 — will be more profitable for the owner of Wacky Quackers in the long run.

(2 marks)

You may write on this page if you need more space to finish your answers to any of the questions in Part 2. Make sure to label each answer carefully (e.g. 12(b)(ii) continued).

Question 16 (12 marks)

Consider the function $f(x) = 4xe^{-x}$ where $x \geq 0$. The graph of $y = f(x)$ is shown below.



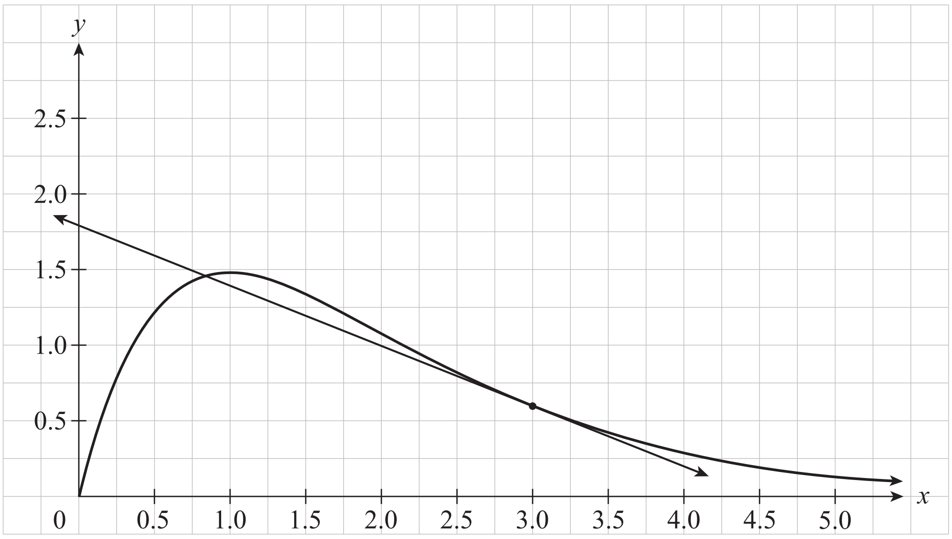
(a) (i) Determine the coordinates of the stationary point of the graph of $y = f(x)$. Give your answer correct to three decimal places.

(1 mark)

(ii) Determine the coordinates of the inflection point of the graph of $y = f(x)$. Give your answer correct to three decimal places.

(2 marks)

The graph of $y = f(x)$ is shown below, along with the tangent to this graph at the point where $x = 3$.



(b) On the graph above, draw a tangent that has a greater y-intercept than that of the tangent shown.

(1 mark)

(c) Show that the tangent to the graph of $y = f(x)$ at the point where $x = a$ has the equation

$$y = 4(1-a)e^{-a}x + 4a^2e^{-a}.$$

(5 marks)

(d) Using the equation given in part (c), determine the value of a that maximises the y-intercept of the tangent to the graph of $y = f(x)$ at the point where $x = a$.

(3 marks)

You may write on this page if you need more space to finish your answers to any of the questions in Part 2. Make sure to label each answer carefully (e.g. 12(b)(ii) continued).