South Australian Certificate of Education

1

# Mathematical Methods

## 2020

## Question booklet 1

- Questions 1 to 6 (52 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on pages 13 and 17 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

## Examination information

**Materials**
- Question booklet 1
- Question booklet 2
- Formula sheet
- SACE registration number label

**Instructions**
- Show appropriate working and steps of logic in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams and graphical representations

**Total time:** 130 minutes
**Total marks:** 100



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## Question 1 (9 marks)

(a) For each of the functions below, determine $f'(x)$. You do not need to simplify your answers.

(i) $f(x) = (2x^5 + 7)^3$.

(2 marks)

(ii) $f(x) = 3\ln(8x+1) + 9\sqrt{x}$.

(2 marks)

(iii) $f(x) = \dfrac{11-5x}{1+\cos x}$.

(3 marks)

(b) Determine the following integral:

$$\int \left(e^{4-3x} + x^2\right) \mathrm{d}x.$$

(2 marks)

**Question 2** (7 marks)

Figure 1 shows the graph of $y = f(x)$ where $f(x) = 2xe^{-x^2}$ for $x \geq 0$.

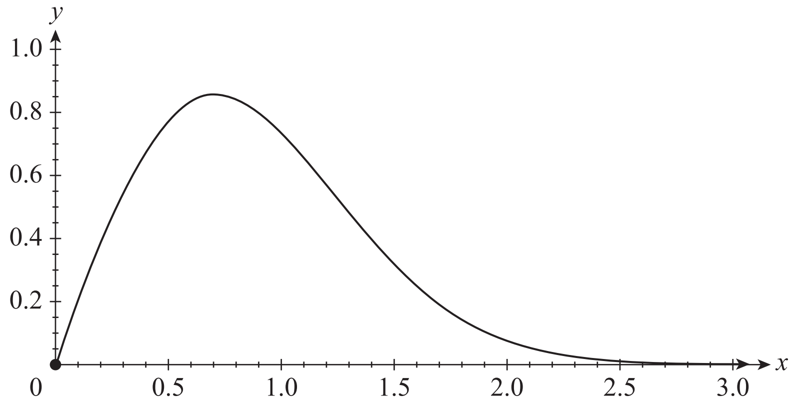



**Figure 1**

The function $f(x)$ is a probability density function, and hence the area of the region between the graph of $y = f(x)$ and the $x$-axis for $x \geq 0$ is equal to $1$.

(a) State *one* other condition that $f(x)$ must satisfy, given that it is a probability density function.

(1 mark)

(b) Calculate $\Pr(0 \leq X \leq 1)$.

(1 mark)

(c) Find $f'(x)$.

(2 marks)

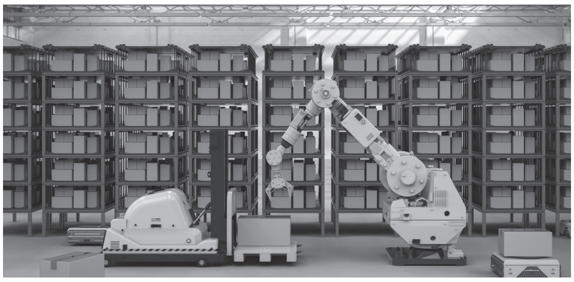
(d) The 'mode' of the probability density function $f(x)$ is the $x$-coordinate at which $f(x)$ is at its maximum value.

Using an algebraic approach, find the *exact* value of the mode of $f(x)$.

(3 marks)

**Question 3** (7 marks)

A particular warehouse uses robotic workers initially to retrieve all products that are requested by customers. However, the probability that a robotic worker retrieves an incorrect product is $10\%$. A human worker is then required to retrieve the correct product.



*Source:* adapted from © PhonlamaiPhoto | iStockphoto.com

(a) Let $X$ be the number of times, from $600$ randomly selected customer requests, that a human worker is required to retrieve the correct product.

(i) State *one* condition that needs to be true in order for $X$ to be modelled by a binomial distribution.

(1 mark)

Assume that $X$ can be modelled by a binomial distribution.

(ii) Calculate the expected number of times, from $600$ randomly selected customer requests, that a human worker will be required to retrieve the correct product.

(1 mark)

(iii) Determine the probability that, from $600$ randomly selected customer requests, a human worker will be required more than $80$ times to retrieve the correct product.

(2 marks)

(b) After a robotic worker has retrieved an incorrect product, a human worker always retrieves the correct product on the first attempt.

The time taken to retrieve a product is, on average:

- $7$ minutes for a robotic worker
- $12$ minutes for a human worker.

Let $T$ be the random variable that represents the *total* time taken (in minutes) to retrieve the correct product. A simple model for $T$ is shown in the discrete probability distribution table below.

| | *Robotic worker retrieves correct product* | *Human worker retrieves correct product* |
|---|---|---|
| *Total time taken (t)* | $7$ | $19$ |
| *Probability* $\Pr(T = t)$ | $0.90$ | $0.10$ |

(i) Explain how the value of $19$ in the table above was obtained, in the context of the problem.

(1 mark)

(ii) Calculate the expected value of the *total* time taken to retrieve the correct product, using the information in the table above.

(1 mark)

(c) One statistician claims that an improved model for $T$ can be developed using a probability density function.

Justify the statistician's claim.

(1 mark)

**Question 4** (8 marks)

Figure 2 shows the graph of $y = f(x)$, where $f(x) = \dfrac{9}{2x^2+1}$.

A chord is shown on the graph between points $A_1\left(-\dfrac{1}{4}, 8\right)$ and $B(1, 3)$.

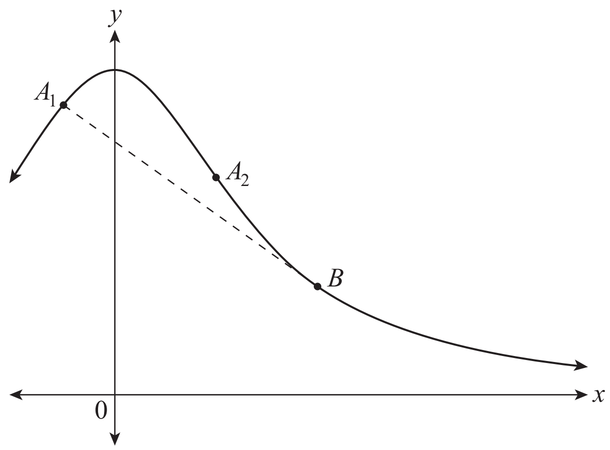



**Figure 2**

(a) Find the slope of chord $A_1B$.

(1 mark)

(b) Using first principles, find the slope of the tangent to the graph of $y = f(x)$ at $B$.

(5 marks)

If a chord was drawn between $A_2$ and $B$, chord $A_2B$ would have a slope of $-6$.

A student makes the following statement:

'On a curved function $g(x)$, the slope of chord $AB$ always approaches the slope of the tangent to $g(x)$ at $B$ when $A$ is moved closer to $B$'.

(c) By considering the slopes of chords $A_1B$ and $A_2B$, and the slope of the tangent to the graph of $y = f(x)$ at $B$, discuss the accuracy of this statement.

(2 marks)

**Question 5** (6 marks)

Figure 3 shows the graph of $y = f(x)$, where

$$f(x) = \frac{1}{2}e^{3-x} - \frac{1}{2}e^{x-3} + 6.$$

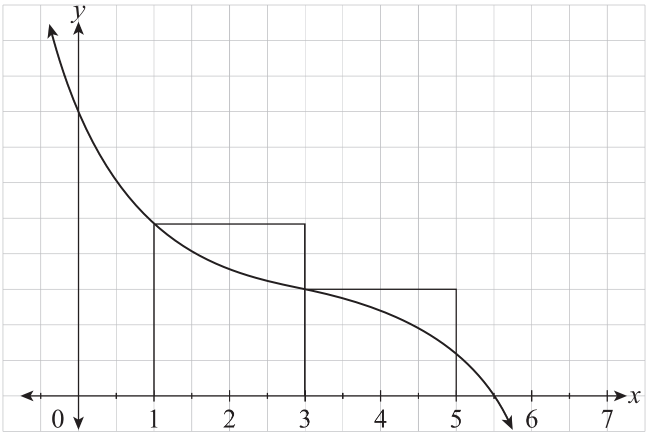



**Figure 3**

Estimates can be used to approximate $A$, the area bounded by $y = f(x)$, the $x$-axis, and the vertical lines $x = 1$ and $x = 5$.

(a) An upper estimate, $U$, of $A$ can be calculated using the areas of two rectangles of equal width, as shown on Figure 3.

Find the *exact* value of $U$.

(3 marks)

A lower estimate, $L$, of $A$ can be calculated using the areas of two rectangles of equal width, as shown on Figure 4.


**Figure 4**

(b) The *exact* value of $L$ is $e^{-2} - e^{2} + 24$.

Show that the average of $U$ and $L$ is 24.

(1 mark)

(c) Write an integral expression to find the value of $A$.

(1 mark)

***Question 5 continues on page 12.***

It is known that:

- $A$ (the area bounded by $y = f(x)$, the $x$-axis, and the vertical lines $x = 1$ and $x = 5$) is 24 square units
- the average of $U$ (the upper estimate) and $L$ (the lower estimate) is 24 square units.

The areas $D_1$, $D_2$, $D_3$, and $D_4$ are shown shaded on Figure 5, where

- $D_1 + D_2$ is the difference between $U$ and $A$
- $D_3 + D_4$ is the difference between $L$ and $A$.

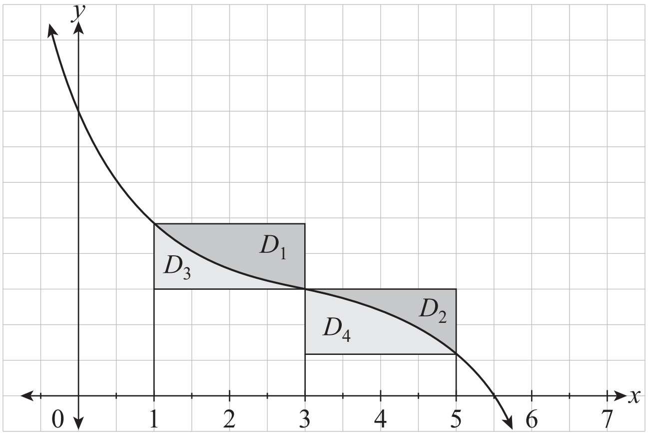



**Figure 5**

(d) State a relationship between $D_1$, $D_2$, $D_3$, and $D_4$ if both $A$, and the average of $U$ and $L$, are equal to 24.

(1 mark)

*You may write on this page if you need more space to finish your answers to any of the questions in Question booklet 1. Make sure to label each answer carefully (e.g. 4(b) continued).*



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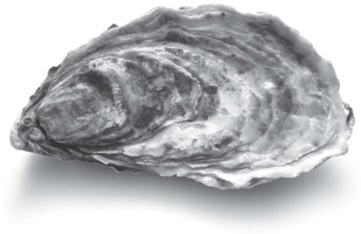
**Question 6** (15 marks)

An oyster farm in South Australia produces oysters to sell to restaurants.

The weight of individual oysters produced by the farm varies, and can be modelled by a normal distribution with a mean of $\mu = 82.3$ grams and a standard deviation of $\sigma = 5.3$ grams.



*Source:* adapted from © julichka | iStockphoto.com

(a) Calculate the probability that a randomly selected oyster from this farm weighs between $80$ grams and $90$ grams.

(1 mark)

(b) The farm categorises an oyster that weighs $85$ grams or more as a luxury-sized oyster.

(i) Calculate the probability that a randomly selected oyster from this farm is a luxury-sized oyster.

(1 mark)

(ii) Exactly half of the luxury-sized oysters from this farm weigh more than $k$ grams.

Find the value of $k$.

(2 marks)

The farm packs oysters by the dozen $(12)$ to be sold to Australian and European restaurants. Each of the $12$ oysters in a package is randomly selected.

When sold to European restaurants, a package of $12$ oysters has a labelled net weight of $1$ kilogram. ($1$ kilogram $= 1000$ grams.)

(c) Consider $S_{12}$, the net weight of a package of $12$ oysters.

(i) State the mean and standard deviation of $S_{12}$.

(2 marks)

(ii) Determine the proportion of packages of $12$ oysters that have a net weight of less than $1$ kilogram.

(1 mark)

(iii) Hence is it appropriate for this farm to sell its packages of $12$ oysters that have a labelled net weight of $1$ kilogram to European restaurants? Justify your answer.

(2 marks)

***Question 6 continues on page 16.***

(d) The mean weight of individual oysters produced by the farm was originally $82.3$ grams.

The farm introduces a new farming method to produce oysters that have a greater individual weight. A sample of $100$ oysters produced using this new farming method is randomly selected. The mean weight of the oysters in the sample is $89.1$ grams.

(i) Using the sample data, calculate a $95\%$ confidence interval for the mean weight of oysters that are produced using the new farming method. Assume that the standard deviation remains at $\sigma = 5.3$ grams.

(2 marks)

(ii) Does the confidence interval that you calculated suggest that the new farming method has increased the mean weight of oysters produced by this farm? Justify your answer.

(1 mark)

(iii) After introducing the new farming method, the farm makes the following claim with $95\%$ confidence:

> 'No more than $0.2\%$ of our packages of $12$ oysters will have a net weight that is less than $1$ kilogram'.

Using your answer to part (d)(i), provide mathematical calculations to support the farm's claim.

(3 marks)

*You may write on this page if you need more space to finish your answers to any of the questions in Question booklet 1. Make sure to label each answer carefully (e.g. 6(c)(iii) continued).*

South Australian Certificate of Education

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# Mathematical Methods
## 2020

## Question booklet 2

- Questions 7 to 10 (48 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on pages 7 and 16 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below



**Copy the information from your SACE label here**

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|---|---|---|---|
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**Graphics calculator**

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**Question 7** (9 marks)

Figure 6 shows the graphs of $y = f(x)$ and $y = g(x)$ for $0 \le x \le 7$.

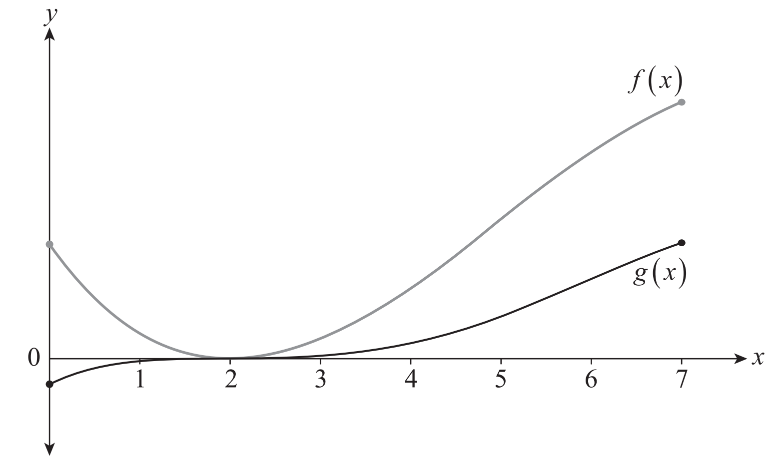


**Figure 6**

The graph of each function has exactly one stationary point; it is located at $x = 2$.

(a) State the nature of the stationary point for the graph of $y = f(x)$ at $x = 2$.

(1 mark)

(b) Figure 7 shows the sign diagram for the second derivative of the function $g(x)$.


**Figure 7**

State the nature of the point that is located at $x = 2$ on the graph of $y = g(x)$.

(1 mark)

(c) The graph of $y = f(x)$ in Figure 6 has one point of inflection at $x = 5$.

Determine the interval(s) for which $f'(x)$ is increasing.

(1 mark)

(d) (i) For $0 < x < 2$, which *one* statement is true? Tick the appropriate box.

$f'(x) < g'(x)$ ☐ $f'(x) = g'(x)$ ☐ $f'(x) > g'(x)$ ☐ (1 mark)

(ii) For $0 < x < 2$, which *one* statement is true? Tick the appropriate box.

$f''(x) < g''(x)$ ☐ $f''(x) = g''(x)$ ☐ $f''(x) > g''(x)$ ☐ (1 mark)

(iii) For $2 < x < 7$, given that the value of $f(x) - g(x)$ is increasing, which *one* statement is true? Tick the appropriate box.

$f'(x) < g'(x)$ ☐ $f'(x) = g'(x)$ ☐ $f'(x) > g'(x)$ ☐ (1 mark)

(e) Figure 8 shows the graph of $y = f'(x)$ for $0 \le x \le 7$.

On the axes in Figure 8, sketch a graph of $y = g'(x)$ for $0 \le x \le 7$.

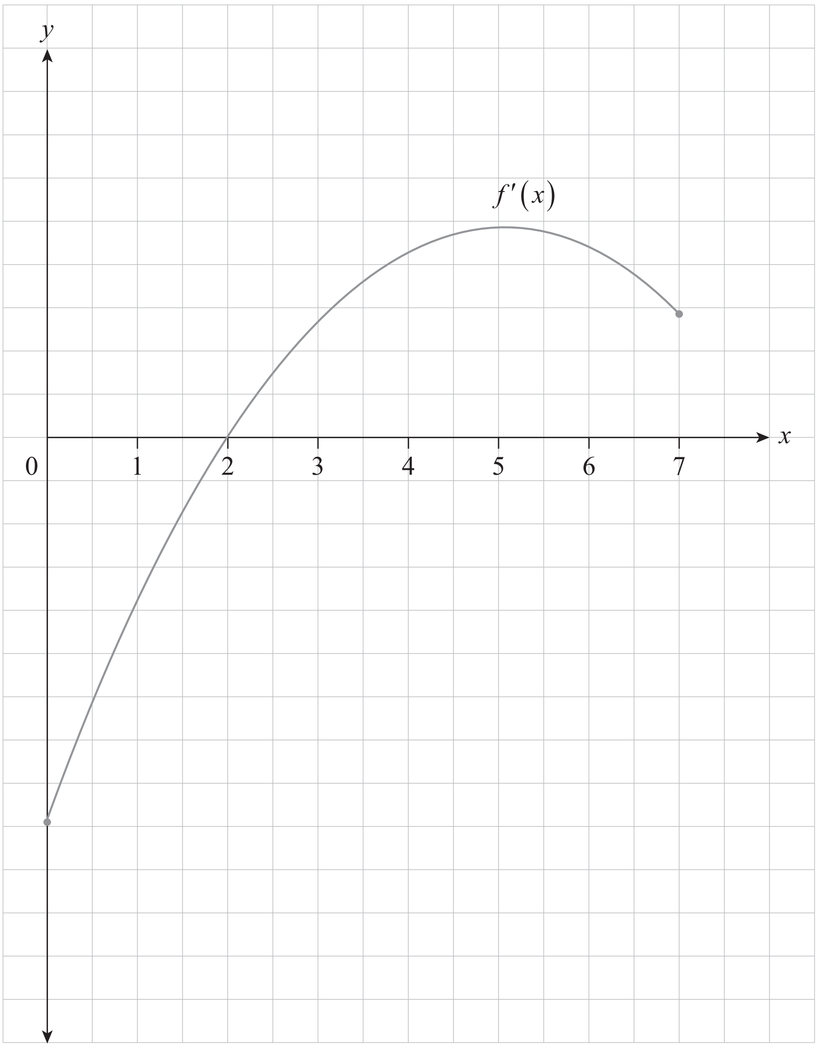


**Figure 8**

(3 marks)

**Question 8** (10 marks)

Figure 9 shows the graph of $y = x\ln\left(x^2+2\right)$. The tangent to the graph at $x = 1$ is also shown.

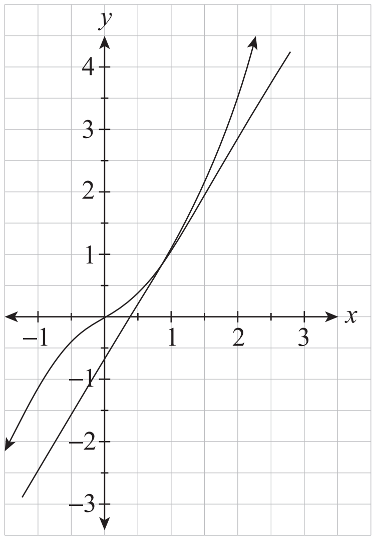


**Figure 9**

(a) (i) Show that $\dfrac{dy}{dx} = \ln\left(x^2+2\right) + \dfrac{2x^2}{x^2+2}$.

(1 mark)

(ii) Hence show that the tangent to the graph of $y = x\ln\left(x^2+2\right)$ at $x = 1$ has equation $y = \left(\ln 3 + \dfrac{2}{3}\right)x - \dfrac{2}{3}$.

(3 marks)

(iii) Determine the $y$-intercept of this tangent.

(1 mark)

Consider the family of functions of the form $y = x\ln\left(x^n + n\right)$ where $n > 0$.

The table below shows the values of the $y$-intercept of the tangent to the graphs of $y = x\ln\left(x^n + n\right)$ at $x = 1$, where $n = 3, 4,$ and $5$.

| $n$ | *Function* | *$y$-intercept of the tangent to the graph of the function at $x = 1$* |
|---|---|---|
| $3$ | $y = x\ln\left(x^3 + 3\right)$ | $-\frac{3}{4}$ |
| $4$ | $y = x\ln\left(x^4 + 4\right)$ | $-\frac{4}{5}$ |
| $5$ | $y = x\ln\left(x^5 + 5\right)$ | $-\frac{5}{6}$ |

(b) Make a conjecture about the value of the $y$-intercept of the tangent to the graph of $y = x\ln\left(x^n + n\right)$ at $x = 1$.

(1 mark)

***Question 8 continues on page 6.***

(c) Prove or disprove the conjecture that you made in part (b) for the $y$-intercept of the tangent to the graph of $y = x\ln\left(x^n + n\right)$ at $x = 1$.

(4 marks)

*You may write on this page if you need more space to finish your answers to any of the questions in Question booklet 2. Make sure to label each answer carefully (e.g. 8(a)(ii) continued).*



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## Question 9 (16 marks)

Figure 10 shows a shaded region that is bounded by the graphs of the functions $y = a(x)$ and $y = b(x)$. These functions intersect at the points $(0, 0)$ and $(6, 6)$. The shaded region is contained within the square formed by the $x$-axis, the $y$-axis, $x = 6$, and $y = 6$.

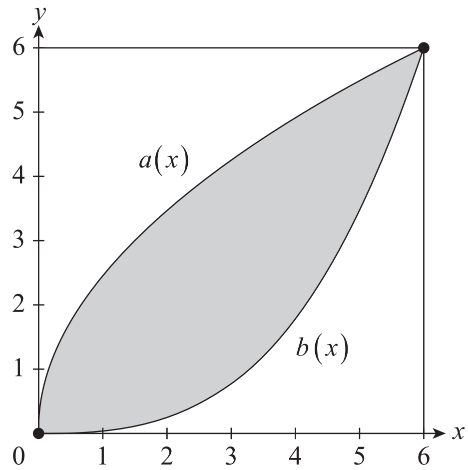


**Figure 10**

A Monte Carlo experiment can be used to estimate the proportion of the square that is shaded, and hence to estimate the area of the shaded region. The experiment uses the following steps:

1. independently randomly generate a sample of $n$ points within the square containing the shaded region; each point within the square has an equal probability of being generated
2. count the number of these points, $x$, that are within the shaded region
3. calculate $\hat{p} = \dfrac{x}{n}$, the proportion of the sample of points that is within the shaded region.

The proportion of the sample of points that is within the shaded region can be used to estimate $A$, the area of the shaded region, using

$$\hat{p} \times \text{area of the square.}$$

(a) A Monte Carlo experiment is conducted using three independently randomly generated points. The results are shown in Figure 11.

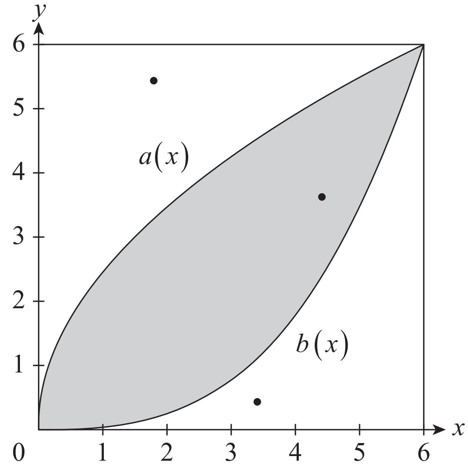


**Figure 11**

(i) Calculate $\hat{p}$, the proportion of the sample of points that is within the shaded region.

(1 mark)

(ii) Hence show that the results from this Monte Carlo experiment indicate that $A$ is estimated to be $12$ square units.

(1 mark)

(iii) State why it may be inappropriate to construct a confidence interval for the proportion of the square that is shaded when using a Monte Carlo experiment involving three independently randomly generated points.

(1 mark)

(b) A new Monte Carlo experiment is conducted using $1500$ independently randomly generated points within the square in Figure 10. Of these $1500$ points, $600$ are within the shaded region.

(i) Using this information, fill in the boxes below to form a $99\%$ confidence interval for the proportion of the square that is shaded.

$0.4 - \square \times \square \leq p \leq 0.4 + \square \times \square$

(2 marks)

(ii) Hence or otherwise, calculate the $99\%$ confidence interval for the proportion of the square that is shaded.

(1 mark)

(iii) A student used an alternative technique to estimate $A$, and claimed that $A$ is at least $14$ square units.

Provide mathematical evidence that the confidence interval does ***not*** support the student's claim.

(2 marks)

(iv) Let $m$ be the number of points that are within the shaded region when $1500$ points are independently randomly generated within the square.

Find the smallest value for $m$ that produces a $99\%$ confidence interval that would support the student's claim that $A$ is at least $14$ square units.

(3 marks)

It is established that the shaded region shown in Figure 10 on page 8 is bounded by $a(x)=(6x)^{\frac{1}{2}}$ and $b(x)=\frac{x^3}{36}$.

(c) (i) Write an integral expression that can be used to find $A$, the area of the shaded region that is bounded by $a(x)$ and $b(x)$.

(2 marks)

(ii) Hence using an algebraic process, determine $A$.

(3 marks)

**Question 10** (13 marks)

*Source:* adapted from © CreativeNature_nl | iStockphoto.com

The rate of change of the number of cars that travel through a particular intersection, per hour, on a typical Monday can be modelled by the function

$$C(t) = -25\cos\left(\frac{\pi}{12}t\right) - 30\cos\left(\frac{\pi}{6}t\right) + 65,$$

where $t$ represents the number of hours since the beginning of Monday.

Figure 12 shows the graph of the model $y = C(t)$ for $0 \le t \le 24$.

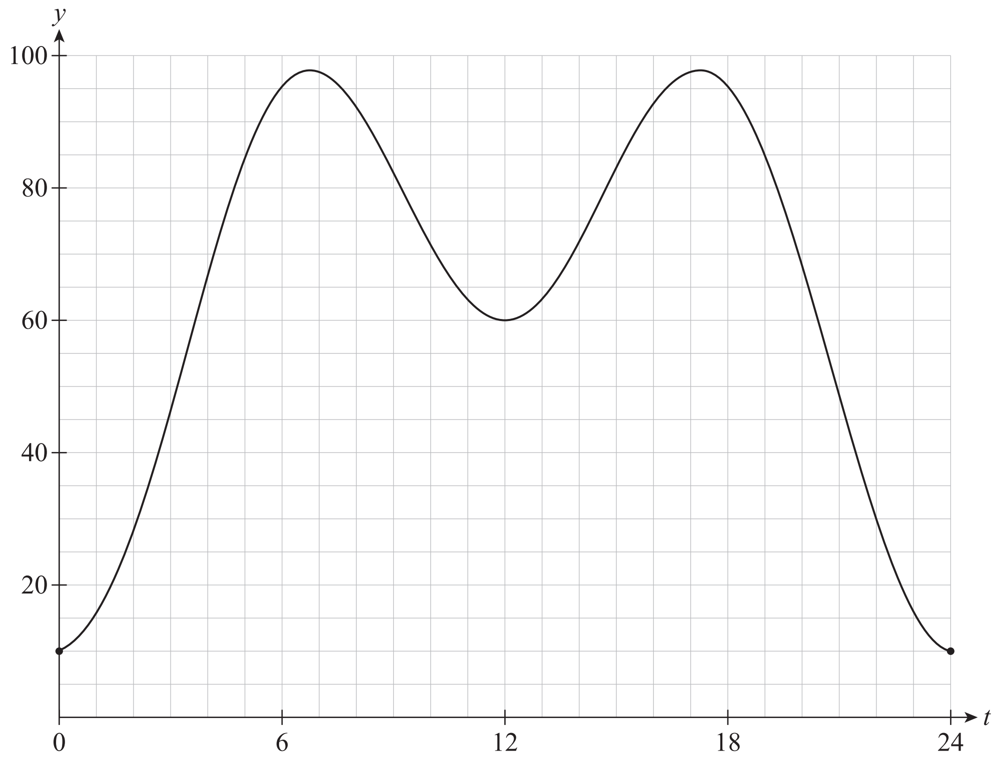


**Figure 12**

(a) Use the mathematical model to predict, for a Monday:

(i) the rate of change of the number of cars that travel through the intersection at $5$ am (i.e. where $t = 5$).

(1 mark)

(ii) the value(s) of $t$ for which the rate of change of the number of cars that travel through the intersection will be at a maximum.

(1 mark)

(iii) the total number of cars that will travel through the intersection between $7$ am and $1$ pm.

(2 marks)

(b) On the axes in Figure 12, represent the quantity that you calculated in part (a)(iii). (1 mark)

***Question 10 continues on page 14.***

(c) The rate of change of the number of cars that travel through the intersection per hour is approximately the same on Mondays and Tuesdays. Figure 13 shows the graph of the model $y = C(t)$ for $0 \le t \le 48$. This model represents the rate of change of the number of cars that travel through the intersection per hour on a typical Monday and Tuesday.

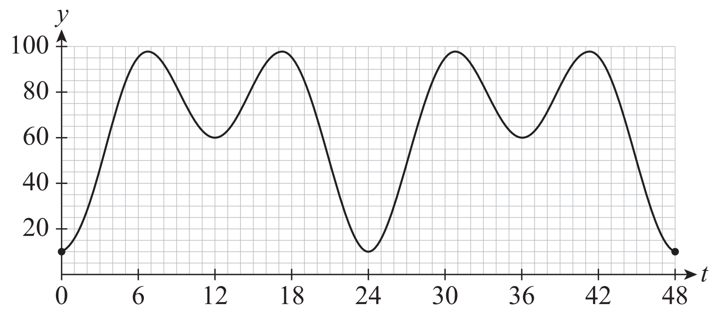


**Figure 13**

The number of cars that travel through the intersection during any 12-hour period commencing at $t = a$ can be modelled by the function

$$N(a) = \int_{a}^{a+12} \left( -25\cos\left(\frac{\pi}{12}t\right) - 30\cos\left(\frac{\pi}{6}t\right) + 65 \right) dt.$$

Maintenance is needed on the intersection. During the maintenance, cars will be unable to travel through the intersection. This maintenance takes 12 hours to complete and must commence at $t = a$ on a Monday, hence $0 \le a \le 24$.

(i) Evaluate the integral above to show that $N(a) = \frac{600}{\pi}\sin\left(\frac{\pi}{12}a\right) + 780$.

*Note:* $\sin(bx + \pi) = -\sin(bx)$ and $\sin(bx + 2\pi) = \sin(bx)$, where $b$ is a real constant.

(4 marks)

(ii) Hence using an algebraic process, find the time on a Monday at which the maintenance should commence in order to minimise the number of cars that will be unable to travel through the intersection.

(4 marks)

*You may write on this page if you need more space to finish your answers to any of the questions in Question booklet 2. Make sure to label each answer carefully (e.g. 10(c)(i) continued).*