South Australian
Certificate of Education

1

Mathematical Methods
2021

Question booklet 1

- Questions 1 to 6 (50 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on page 15 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

Examination information

Materials
- Question booklet 1
- Question booklet 2
- Formula sheet
- SACE registration number label

Instructions
- Show appropriate working and steps of logic in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams and graphical representations

Total time: 130 minutes
Total marks: 100

Attach your SACE registration number label here

Graphics calculator

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Question 1 (7 marks)

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each of the following functions. There is no need to simplify your answers.

(a) $y = 7x^2 + e^{x^2+3x}$

(2 marks)

(b) $y = 3\left(x + \sqrt{x}\right)^5$

(2 marks)

(c) $y = \frac{\cos(2x+9)}{x^2}$

(3 marks)

Question 2 (8 marks)

Consider the function $f(x) = 2x^4 - 8x + 1$. The graph of the function $y = f(x)$ is shown in Figure 1.

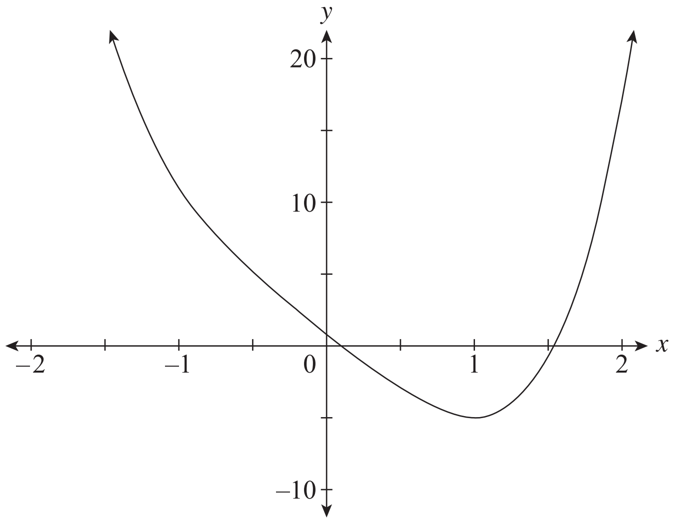

Figure 1

(a) (i) Find $f'(x)$.

(1 mark)

(ii) Hence, using an algebraic process, determine the x-coordinate and the y-coordinate of the stationary point of the graph of $y = f(x)$.

(3 marks)

(b) (i) Find $f''(x)$.

(1 mark)

(ii) Determine whether the graph of $y = f(x)$ has any points of inflection. Justify your answer using mathematical reasoning.

(3 marks)

Question 3 (7 marks)

A new technique of collecting donations for charities is through a mobile phone app. Every time a customer purchases goods via the app, the amount donated is the difference between the total price and the next whole dollar value.

For example, if a customer purchases goods costing \$3.20, their donation will be

$$\$4 - \$3.20 = \$0.80.$$

Let V represent the value, in dollars, of individual donations by customers using the app. The distribution of V has a mean of $\mu_V = \$0.46$ and a standard deviation of $\sigma_V = \$0.27$.

The histogram in Figure 2 represents the distribution of V.

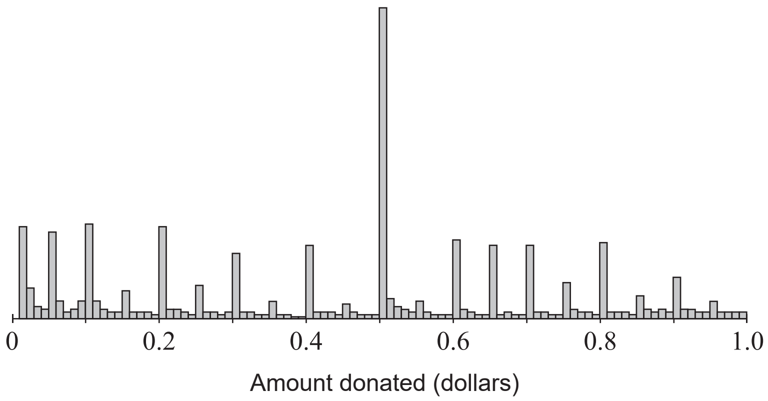

Figure 2

(a) State *one* reason why modelling V with a normal distribution would ***not*** be appropriate.

(1 mark)

Let S_n represent the sum of n randomly selected donations. The two histograms in Figures 3 and 4 represent the distributions of S_2 and S_{10}.

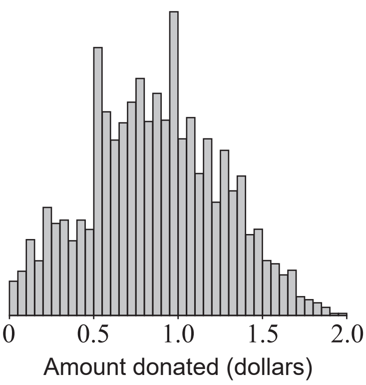

Figure 3

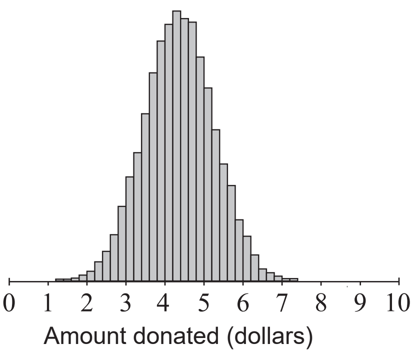

Figure 4

(b) State which histogram — Figure 3 or Figure 4 — represents the distribution of S_{10}. Justify your answer.

(2 marks)

(c) (i) Hence or otherwise, explain why it would be reasonable to assume that S_{900} can be modelled by a normal distribution.

(1 mark)

(ii) Show that the distribution of S_{900} has a mean of $\$414$ and a standard deviation of $\$8.10$.

(2 marks)

(iii) A charity receives 900 random donations from the app on a given day.

Calculate the probability that the charity receives at least $\$400$ from the 900 donations.

(1 mark)

Question 4 (7 marks)

Consider the functions $p(x) = \ln x$ and $q(x) = \ln 3x$.

Figure 5 shows the graphs of $y = p(x)$ and $y = q(x)$.

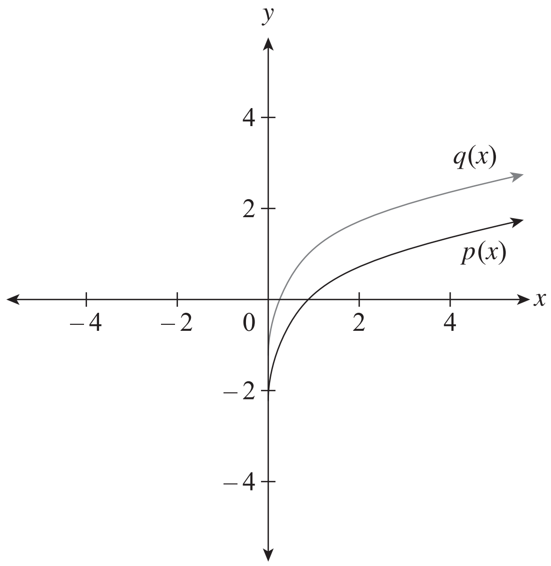

Figure 5

(a) Show that $p'(x) = q'(x)$.

(2 marks)

(b) Describe the relationship between the graphs of $y = p(x)$ and $y = q(x)$.

(1 mark)

Consider the function $r(x) = \ln(x^2 + 6x + 9)$ with domain $x > -3$.
Figure 6 shows the graphs of $y = p(x)$ and $y = r(x)$.

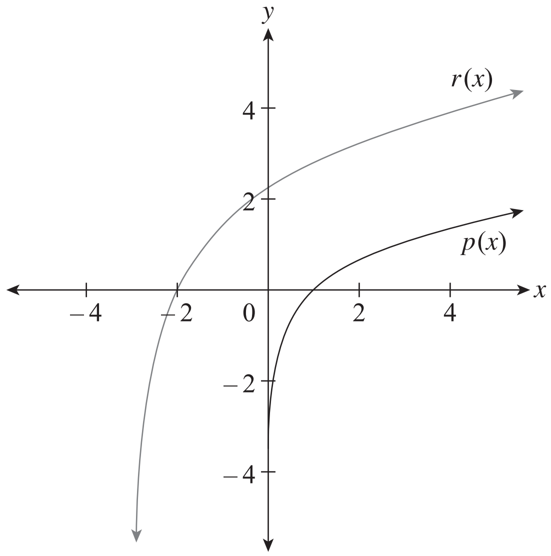

Figure 6

(c) The function $r(x)$ can be written in the form $a\ln(x+b)$, where a and b are positive integers.

(i) Determine the values of a and b.

(2 marks)

(ii) Hence, describe the relationship between the graphs of $y = p(x)$ and $y = r(x)$.

(2 marks)

Question 5 (9 marks)

A large number of guests of a hotel are required to anonymously rate their stay from 1 star to 5 stars. The best possible rating is 5 stars.

The distribution of the rating, X, given to the hotel by guests is shown in the table below.

Source: adapted from © Tero Vesalainen | dreamstime.com

x	1	2	3	4	5
$\Pr(X = x)$	0.045	0.015	0.105	0.282	a

(a) Show that $a = 0.553$.

(1 mark)

(b) Determine the mean rating given to the hotel by its guests.

(1 mark)

(c) Show that the probability that a randomly selected guest will rate the hotel less than 3 stars is 0.06.

(1 mark)

Let Y represent the number of guests from a random sample who rated the hotel less than 3 stars.
Y can be modelled using a binomial distribution.

(d) The hotel's managers randomly select a group of 20 guests. Calculate the probability that:

(i) exactly *two* guests rate the hotel less than 3 stars.

(1 mark)

(ii) more than *five* guests rate the hotel less than 3 stars.

(2 marks)

(e) Managers are concerned that the hotel is receiving ratings of less than 3 stars and they decide to interview past guests.

How large a group of randomly selected guests is needed for the managers to have a greater than 90% chance of having at least *one* guest who has rated the hotel less than 3 stars?

(3 marks)

PLEASE TURN OVER

Question 6 (12 marks)

The concentration of caffeine in the blood plasma of an adult t hours after they have consumed a 150 milligram dose of caffeine can be modelled by the function

$$c(t)=15\left(e^{-0.3t}-e^{-0.6t}\right) \text{ for } t\geq 0,$$

where the concentration, $c(t)$, is measured in milligrams per litre (mg L^{-1}).

Source: adapted from
© Bagwold | dreamstime.com

(a) *On the set of axes in Figure 7*, sketch a graph of $y=c(t)$.

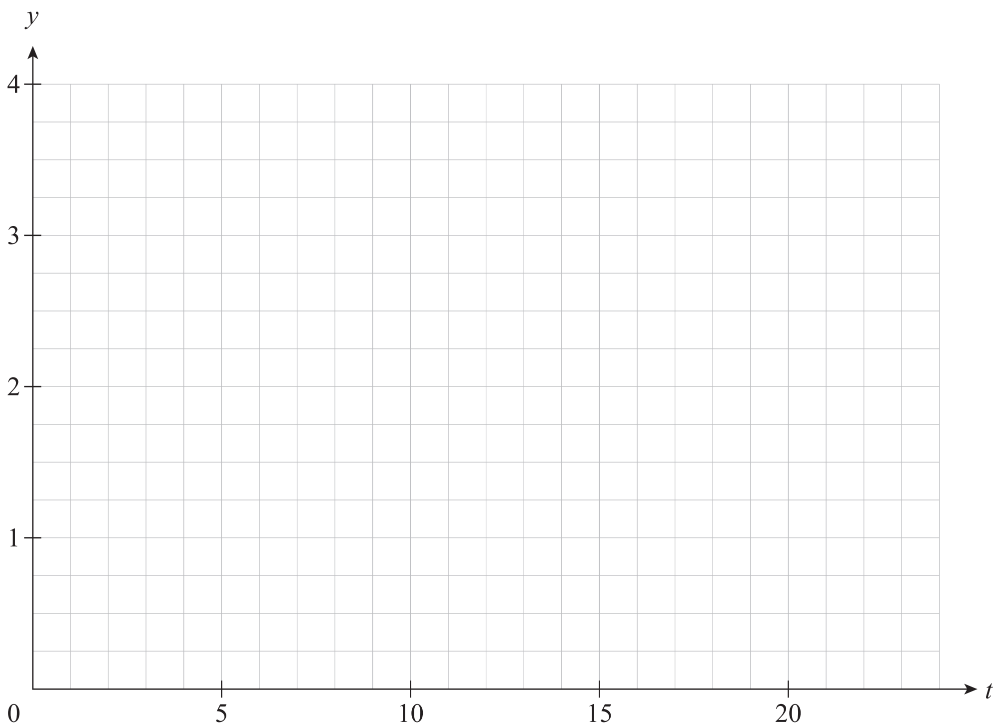

Figure 7

(2 marks)

(b) Using the model above, determine the concentration of caffeine in the blood plasma of an adult 10 hours after they have consumed a 150 milligram dose of caffeine.

(1 mark)

(c) (i) Determine $c'(2)$.

(1 mark)

(ii) Interpret the value of $c'(2)$ in the context of the problem.

(2 marks)

(d) (i) Show that $c'(t) = -4.5e^{-0.3t} + 9e^{-0.6t}$.

(1 mark)

(ii) Hence, by letting $c'(t) = 0$, show that the concentration of caffeine in an adult's blood plasma reaches its maximum at $t = \frac{10}{3}\ln 2$ hours.

(3 marks)

Question 6 continues on page 14.

PLEASE TURN OVER

A general model for the concentration, $c_d(t)$, of caffeine in the blood plasma of an adult t hours after they have consumed a dose of d milligrams of caffeine is

$$c_d(t) = \frac{d}{10}\left(e^{-0.3t} - e^{-0.6t}\right) \text{ for } t \geq 0,$$

where $c_d(t)$ is measured in milligrams per litre (mg L^{-1}).

The maximum concentration of caffeine in an adult's blood plasma in the general model also occurs at $t = \frac{10}{3}\ln 2$ hours.

(e) If the concentration of caffeine in an adult's blood plasma is greater than 15 mg L^{-1}, the adult will experience serious side effects.

Show that the general model predicts that a dose of 600 milligrams of caffeine is the maximum an adult can consume without experiencing serious side effects.

(2 marks)

You may write on this page if you need more space to finish your answers to any of the questions in Question booklet 1. Make sure to label each answer carefully (e.g. 6(d)(ii) continued).

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Mathematical Methods
2021

Question booklet 2

- Questions 7 to 11 (50 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on pages 7 and 15 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

2

Copy the information from your SACE label here

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Question 7 (8 marks)

Consider the function $f(x) = 8 - 2^{0.5x}$. The graph of $y = f(x)$ is shown in Figure 8.

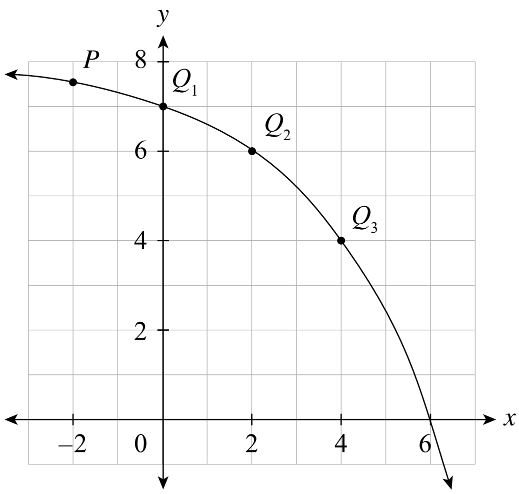


Figure 8

Points P, Q_1, Q_2, and Q_3 are shown on the graph in Figure 8. Their coordinates are: $P\,(-2, 7.5)$, $Q_1\,(0, 7)$, $Q_2\,(2, 6)$, and $Q_3\,(4, 4)$.

(a) Chords can be drawn from P to each of Q_1, Q_2, or Q_3.

(i) Identify which of the following chords has a slope that would provide the best approximation for the slope of the tangent to the graph of $y = f(x)$ at P. Tick the appropriate box to indicate your answer.

PQ_1 ☐ PQ_2 ☐ PQ_3 ☐

(1 mark)

(ii) Calculate the slope of the chord that you selected in part (a)(i).

(2 marks)

(b) (i) Show that $f(x)$ can be expressed as $f(x) = 8 - e^{(0.5\ln 2)x}$.

(1 mark)

(ii) Hence, show that $f'(x) = -0.5\ln 2 \times 2^{0.5x}$.

(1 mark)

(c) Using an algebraic process, find the *exact* equation of the tangent to the graph of $y = f(x)$ at *P*.

(3 marks)

Question 8 (10 marks)

Consider the continuous random variable X, with the probability density function $f(x) = \frac{1}{8}x$ for $0 \le x \le 4$. A graph of $y = f(x)$ is shown in Figure 9.

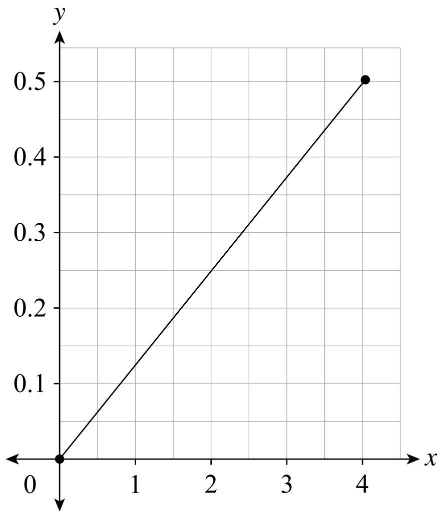


Figure 9

(a) Given $\mu_X = \frac{8}{3}$, calculate σ_X.

(2 marks)

(b) Find $\Pr(2 \le X \le 3)$.

(1 mark)

(c) Using integration and an algebraic process, show that $\Pr\left(0 \le X \le 1\right) = \frac{1}{16}$.

(2 marks)

Consider the real numbers m and n, such that $\Pr\left(m \le X \le n\right) = \frac{1}{16}$ where $0 \le m \le 4$ and $0 \le n \le 4$.

The following conjecture is made for the value of n in terms of m:

$$n = \sqrt{m^2 + 1}.$$

(d) Prove this conjecture.

(3 marks)

(e) Use the conjecture $n=\sqrt{m^2+1}$ to determine the *exact* maximum value of m that satisfies the probability statement $\Pr(m \le X \le n)=\frac{1}{16}$, for $0 \le x \le 4$.

(2 marks)

You may write on this page if you need more space to finish your answers to any of the questions in Question booklet 2. Make sure to label each answer carefully (e.g. 8(d) continued).

PLEASE TURN OVER

Question 9 (8 marks)

A small number of red tokens and a large number of blue tokens are placed in a bag.

(a) A student randomly draws 50 tokens with replacement, and 6 of these tokens are red.

Calculate a 99% confidence interval for the proportion of red tokens in the bag.

(2 marks)

(b) The lower bound of a 99% confidence interval for a proportion is calculated using the formula

$$\hat{p} - 2.58\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

If the lower bound of a 99% confidence interval for a proportion is calculated to be zero, show that

$$\hat{p} = \frac{2.58^2}{n + 2.58^2} \text{ when } \hat{p} > 0.$$

(3 marks)

If the true proportion of elements in a population that have a given characteristic is close to zero, a confidence interval for this proportion may contain a lower bound that is less than or equal to zero.

For example, if 5 of the 50 tokens randomly drawn were red, instead of 6 as in part (a), the 99% confidence interval for the proportion of red tokens in the bag would have been calculated as:

$$-0.00928 \leq p \leq 0.209.$$

When the lower bound is less than or equal to zero, statisticians would use an alternative process to calculate the confidence interval, so that it does not contain any values for p that are less than or equal to zero.

(c) Assume 1000 tokens are randomly drawn with replacement, and x of these tokens are red. Using the formula

$$\hat{p} = \frac{2.58^2}{n + 2.58^2}$$

determine the lowest possible value of x that would avoid the use of an alternative process to calculate a 99% confidence interval.

(3 marks)

Question 10 (9 marks)

Some functions do not have antiderivatives that can be expressed in terms of standard mathematical functions. For these functions, only an estimate of the area between the graph of the function and the x-axis for a given domain can be found.

Consider one such function: $f(x) = x^x$ for $x > 0$.

Figure 10 shows the graph of $y = f(x)$ and four rectangles of equal width that can be used to calculate an underestimate of the area between the graph of $y = f(x)$ and the x-axis for $1 \le x \le 3$.

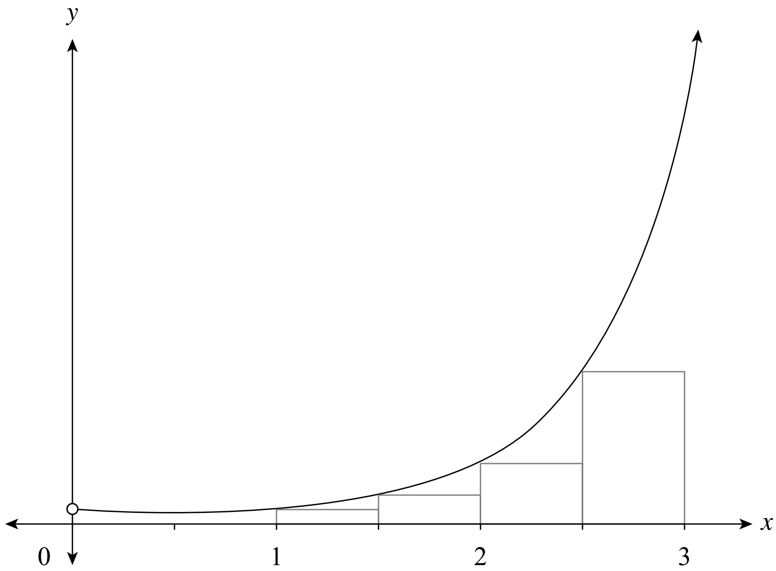


Figure 10

(a) Calculate this underestimate, expressing your answer correct to *five* significant figures.

(2 marks)

(b) *On the set of axes in Figure 10 above*, add four rectangles of equal width that could be used to calculate an overestimate of the area between the graph of $y = f(x)$ and the x-axis for $1 \le x \le 3$.

(1 mark)

(c) (i) Determine $f''(x)$, given that the derivative of $f(x) = x^x$ for $x > 0$ is $f'(x) = x^x(\ln x + 1)$.

(2 marks)

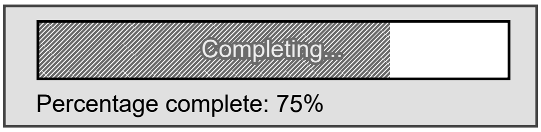
(ii) Hence, show that the graph of $y = f(x)$ for $x > 0$ is always convex (i.e. concave up).

(2 marks)

(iii) Hence, when using rectangles of equal width to approximate the area bounded by the graph of $y = f(x)$ and the x-axis for $1 \le x \le 3$, is it more accurate to use an underestimate or overestimate? Justify your answer.

(2 marks)

Question 11 (15 marks)



Progress bars show computer users the 'percentage complete' of a task. Research has indicated that a user's perceived completion time can be affected by changes in the speed of a progress bar during the completion of a task.

To investigate perceived completion time, researchers create two 30-second tasks. The progress bar for each task accurately displays the completion of the task from 0% to 100% during the 30 seconds.

(a) Task A completes at an increasing rate, resulting in a progress bar that speeds up during the 30 seconds.

The researchers predict that the mean perceived completion time for Task A will be shorter than the actual completion time (30 seconds).

To test this prediction, 100 users were shown the progress bar for Task A and their perceived completion time, x_A, was recorded. For the sample of 100 users, the mean perceived completion time for Task A was $\bar{x}_A = 29.2$ seconds.

(i) Calculate a 95% confidence interval for the mean perceived completion time, μ_A, for Task A. Assume the population standard deviation was $\sigma_A = 7.6$ seconds.

(2 marks)

(ii) Can the researchers' prediction be supported with 95% confidence? Justify your answer.

(2 marks)

(b) Task B pauses during the completion of the task, resulting in a progress bar that stops momentarily during the 30 seconds.

Researchers predict that the mean perceived completion time for Task B will be more than 10% longer than the actual completion time (30 seconds).

To test this prediction, 100 users were shown the progress bar for Task B and their perceived completion time, x_B, was recorded. For the sample of 100 users, the 95% confidence interval for the mean perceived completion time for Task B was:

$$33.6 \le \mu_B \le 36.8.$$

Give evidence that the researchers' prediction about Task B can be supported with 95% confidence.

(2 marks)

(c) The speed of the progress bar (the rate of change of the percentage complete) for Task A was created using the function

$$T_A'(t) = \frac{\sqrt{30t}}{6} \text{ for } 0 \le t \le 30,$$

where t is the number of seconds since the beginning of the task.

(i) Write an integral expression for the percentage complete after 15 seconds.

(1 mark)

(ii) Hence, determine the percentage complete after 15 seconds.

(1 mark)

Question 11 continues on page 14.

(d) The speed of the progress bar (the rate of change of the percentage complete) for Task B was created using the function

$$T_B'(t) = a\cos(bt) + a \text{ for } 0 \le t \le 30,$$

where a and b are positive constants, and t is the number of seconds since the beginning of the task.

(i) Given that Task B pauses only once at $t = 24$ seconds, show that the *only* possible value for b is $\frac{\pi}{24}$.

(4 marks)

(ii) Hence, given that Task B is completed in 30 seconds, determine the *exact* value of a.

(3 marks)

You may write on this page if you need more space to finish your answers to any of the questions in Question booklet 2. Make sure to label each answer carefully (e.g. 11(d)(ii) continued).

MATHEMATICAL METHODS FORMULA SHEET

Properties of derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(f(x)\,g(x)\Big) = f'(x)g(x) + f(x)g'(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$$

Quadratic equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Discrete random variables

The mean or expected value of a discrete random variable is:

$$\mu_X = \sum x.p(x),$$

where $p(x)$ is the probability function for achieving result x.

The standard deviation of a discrete random variable is:

$$\sigma_X = \sqrt{\sum [x - \mu_X]^2 \; p(x)},$$

where μ_X is the expected value and $p(x)$ is the probability function for achieving result x.

Bernoulli distribution

The mean of the Bernoulli distribution is p, and the standard deviation is:

$$\sqrt{p(1-p)}.$$

Binomial distribution

The mean of the binomial distribution is np, and the standard deviation is:

$$\sqrt{np(1-p)},$$

where p is the probability of success in a single Bernoulli trial and n is the number of trials.

The probability of k successes from n trials is:

$$\Pr(X = k) = C_k^n p^k (1-p)^{n-k},$$

where p is the probability of success in the single Bernoulli trial.

Population proportions

The sample proportion is $\hat{p} = \dfrac{X}{n}$,

where a sample of size n is chosen, and X is the number of elements with a given characteristic.

The distribution of a sample proportion has a mean of p and a standard deviation of

$$\sqrt{\frac{p(1-p)}{n}}.$$

The upper and lower limits of a confidence interval for the population proportion are:

$$\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},$$

where the value of z is determined by the confidence level required.

Continuous random variables

The mean or expected value of a continuous random variable is:

$$\mu_X = \int_{-\infty}^{\infty} x\,f(x)\,\mathrm{d}x,$$

where $f(x)$ is the probability density function.

The standard deviation of a continuous random variable is:

$$\sigma_X = \sqrt{\int_{-\infty}^{\infty} [x - \mu_X]^2 f(x)\,\mathrm{d}x},$$

where $f(x)$ is the probability density function.

Normal distributions

The probability density function for the normal distribution with mean μ and standard deviation σ is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

All normal distributions can be transformed to the standard normal distribution with $\mu = 0$ and $\sigma = 1$ by:

$$Z = \frac{X - \mu}{\sigma}.$$

Sampling and confidence intervals

If $\bar{x}$ is the sample mean and s the standard deviation of a suitably large sample, then the upper and lower limits of the confidence interval for the population mean are:

$$\bar{x} - z\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + z\frac{s}{\sqrt{n}},$$

where the value of z is determined by the confidence level required.