Question 14 (15 marks)

The complex number w is shown on the Argand diagram in Figure 14.

The dashed rays represent the complex numbers such that $\arg z = \frac{\pi}{6}$, $\arg z = \frac{\pi}{4}$, and $\arg z = \frac{\pi}{3}$.

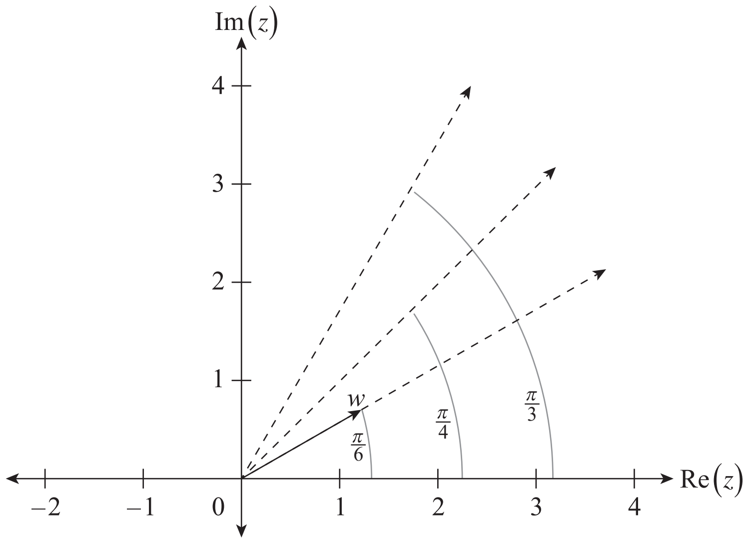

Figure 14

(a) On the Argand diagram in Figure 14:

(i) draw the approximate position of w^3. (1 mark)

(ii) sketch the set of complex numbers z such that $\left|z - w^3\right| = \left|w\right|$. (2 marks)

(b) (i) Write $z = \frac{\sqrt{6}}{2} + \frac{i\sqrt{2}}{2}$ in exact polar form.

(1 mark)

(ii) Use de Moivre's theorem to show that $z^6 = -8$.

(1 mark)

(c) (i) Show that the solutions of $z^6 = -8$ are: $z_1 = \sqrt{2}\text{cis}\left(\frac{\pi}{6}\right)$; $z_2 = \sqrt{2}\text{cis}\left(\frac{\pi}{2}\right)$; $z_3 = \sqrt{2}\text{cis}\left(\frac{5\pi}{6}\right)$;

$z_4 = \sqrt{2}\text{cis}\left(-\frac{5\pi}{6}\right)$; $z_5 = \sqrt{2}\text{cis}\left(-\frac{\pi}{2}\right)$; and $z_6 = \sqrt{2}\text{cis}\left(-\frac{\pi}{6}\right)$.

(2 marks)

On the Argand diagram in Figure 15, P represents $z_1^{\ 3}$ and Q represents $2z_1$, where $z_1 = \sqrt{2}\text{cis}\left(\frac{\pi}{6}\right)$.

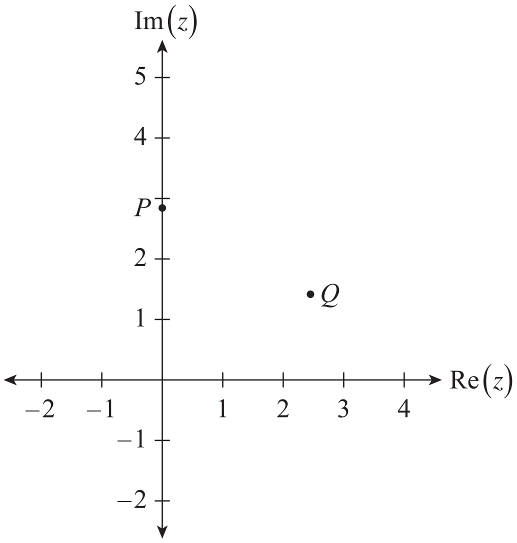

Figure 15

(ii) On the Argand diagram in Figure 15:

(1) sketch $|z| = \sqrt{2}$. (1 mark)

(2) draw and label z_1, z_2, z_3, z_4, z_5, and z_6. (2 marks)

(iii) Find the exact value of $\left| z_1^{\,3} - 2z_1 \right|$.

(2 marks)

Circles of radius $\sqrt{2}$ are drawn with centres P and Q.

(iv) Deduce that these circles touch.

(1 mark)

(v) Find the point where the circles touch, and write it in exact $x + iy$ form.

(2 marks)