Question 9 (11 marks)

Let $f(x) = \sqrt{8 - x^3}$ for $x < 2$.

(a) Show that $f'(x) = \dfrac{-3x^2}{2\sqrt{8 - x^3}}$ for $x < 2$.

$f(x) = (8 - x^3)^{1/2}$

$f'(x) = \frac{1}{2}(8 - x^3)^{-1/2} \cdot -3x^2$

$= \dfrac{-3x^2}{2\sqrt{8 - x^3}}$

(1 mark)

The graph of $y = f(x)$ is shown in Figure 9, along with the normal to the graph at $x = -1$.

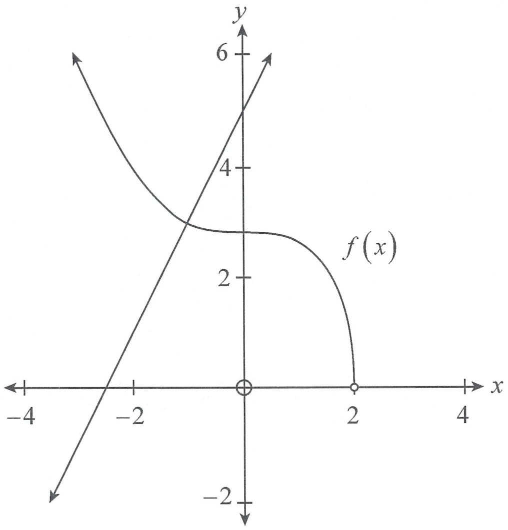

Figure 9

(b) (i) Show that the normal to the graph of $y = f(x)$ at $x = -1$ has the equation $2x - y = -5$.

$f(-1) = \sqrt{8+1} = 3$

$f'(-1) = \dfrac{-3}{2\sqrt{8+1}} = -\frac{1}{2}$

$\therefore$ The gradient of the normal is 2

$\therefore$ The equation of the normal is $y - 3 = 2(x+1)$

$y - 3 = 2x + 2$

$2x - y = -5$

(3 marks)

(ii) State the x-intercept of the normal $2x - y = -5$.

$\left(-\frac{5}{2}, 0\right)$

(1 mark)

(c) Consider the normal to the graph of $y = f(x)$ at $x = a$, where $a < 2$.

(i) Show that the x-intercept of this normal is $x = a - \frac{3}{2}a^2$.

$f(a) = \sqrt{8 - a^3}$

$f'(a) = \dfrac{-3a^2}{2\sqrt{8 - a^3}}$

$\therefore$ The gradient of the normal is $\dfrac{2\sqrt{8 - a^3}}{3a^2}$

$\therefore$ The equation of the normal is $y - \sqrt{8 - a^3} = \dfrac{2\sqrt{8 - a^3}}{3a^2}(x - a)$

Substituting $y = 0$ to find the x-intercept gives $-\cancel{\sqrt{8 - a^3}} = \dfrac{2\cancel{\sqrt{8 - a^3}}}{3a^2}(x - a)$

$x - a = -\dfrac{3a^2}{2}$

$x = a - \dfrac{3a^2}{2}$

(4 marks)

(ii) Hence, using an algebraic approach, find the value of a such that the x-intercept of this normal is maximised.

$\dfrac{dx}{da} = 0 \Rightarrow 1 - 3a = 0$

$3a = 1$

$a = \frac{1}{3}$

Sign of $\frac{dx}{da}$: $+$ $\nearrow$ $\frown$ $-$ $\searrow$ at $a = \frac{1}{3}$

The x-intercept is maximised when $a = \frac{1}{3}$

(2 marks)