Question 10 (16 marks)

(a) (i) State the roots of the complex equation $w^6 = 1$ in $r\,\text{cis}\,\theta$ form.

$z = \text{cis}\left(0 + k.\frac{2\pi}{6}\right)$ where $k \in \mathbb{Z}$

ie. $z_1 = \text{cis}(0) = 1$, $z_2 = \text{cis}\left(\frac{\pi}{3}\right)$, $z_3 = \text{cis}\left(\frac{2\pi}{3}\right)$, $z_4 = \text{cis}(\pi) = -1$, $z_5 = \text{cis}\left(-\frac{2\pi}{3}\right)$, $z_6 = \text{cis}\left(-\frac{\pi}{3}\right)$

(2 marks)

(ii) On the Argand diagram in Figure 10, plot the roots identified in part (a)(i).

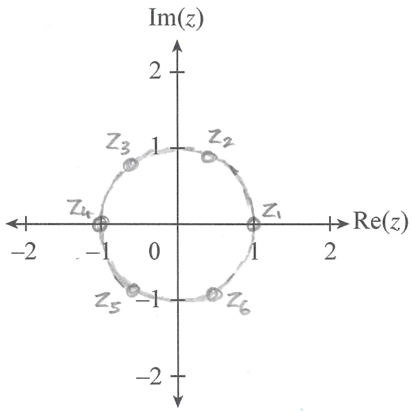

Figure 10

(2 marks)

(b) Consider the complex equation $(z-1)^6 = 1$.

(i) Using the roots identified in part (a)(i), state all the roots of the equation $(z-1)^6 = 1$, giving answers in any form.

$\text{cis}(0)+1$, $\text{cis}\left(\frac{\pi}{3}\right)+1$, $\text{cis}\left(\frac{2\pi}{3}\right)+1$, $\text{cis}(\pi)+1$, $\text{cis}\left(-\frac{2\pi}{3}\right)+1$, $\text{cis}\left(-\frac{\pi}{3}\right)+1$

(2 marks)

(ii) On the Argand diagram in Figure 11, plot the roots of the equation $(z-1)^6 = 1$.

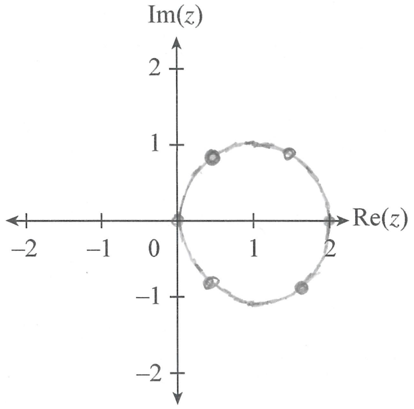

Figure 11

(1 mark)

(iii) Write the roots of $(z-1)^6 = 1$ in $r\,\text{cis}\,\theta$ form or real form.

2, $\frac{3}{2} + \frac{i\sqrt{3}}{2} = \sqrt{3}\,\text{cis}\left(\frac{\pi}{6}\right)$, $\frac{1}{2} + \frac{i\sqrt{3}}{2} = \text{cis}\left(\frac{\pi}{3}\right)$

0, $\frac{1}{2} - \frac{i\sqrt{3}}{2} = \text{cis}\left(-\frac{\pi}{3}\right)$, $\frac{3}{2} - \frac{i\sqrt{3}}{2} = \sqrt{3}\,\text{cis}\left(\frac{-\pi}{6}\right)$

(2 marks)

(c) (i) Suppose that the polynomial $z^2 + bz + c$ has a zero $r\,\text{cis}\,\theta$, where b and c are real, and $r > 0$ and $0 < \theta < \pi$.

Show that $b = -2r\cos\theta$ and $c = r^2$.

If $r\,\text{cis}\,\theta = r(\cos\theta + i\sin\theta)$ is a zero of the polynomial

then $r\,\text{cis}(-\theta) = r(\cos\theta - i\sin\theta)$ is also a zero of the polynomial

The sum of the roots is $2r\cos\theta$

The product of the roots is $r^2(\cos^2\theta + \sin^2\theta) = r^2$

$\therefore$ The polynomial is $z^2 - 2r\cos\theta\, z + r^2$

ie. $b = -2r\cos\theta$ and $c = r^2$

(2 marks)

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(ii) Verify that $(z-1)^6 = z^6 - 6z^5 + 15z^4 - 20z^3 + 15z^2 - 6z + 1$.

$(z-1)^6 = (z-1)^3 (z-1)^3$

$= (z^3 - 3z^2 + 3z - 1)(z^3 - 3z^2 + 3z - 1)$

$= z^6 - 3z^5 + 3z^4 - z^3 - 3z^5 + 9z^4 - 9z^3 + 3z^2 + 3z^4 - 9z^3 + 9z^2 - 3z - z^3 + 3z^2 - 3z + 1$

$= z^6 - 6z^5 + 15z^4 - 20z^3 + 15z^2 - 6z + 1$

(2 marks)

(d) Using part (b)(iii) and part (c), factorise $z^6 - 6z^5 + 15z^4 - 20z^3 + 15z^2 - 6z$ into the product of real linear and real quadratic factors.

$(z-1)^6 = 1 \Rightarrow z^6 - 6z^5 + 15z^4 - 20z^3 + 15z^2 - 6z + 1 = 1$

ie. $z^6 - 6z^5 + 15z^4 - 20z^3 + 15z^2 - 6z = 0$

From (b)(iii), $\sqrt{3}\,\text{cis}\left(\frac{\pi}{6}\right)$ is a zero of this polynomial

$\therefore$ $z^2 - 2\sqrt{3}\cos\left(\frac{\pi}{6}\right)z + 3 = z^2 - 3z + 3$ is a factor of this polynomial

Also from (b)(iii) $\text{cis}\left(\frac{2\pi}{3}\right)$ is a zero of this polynomial

$\therefore$ $z^2 - 2\cos\left(\frac{2\pi}{3}\right)z + 1 = z^2 - z + 1$ is a zero of this polynomial

Also from (b)(iii) 0 and 2 are zeros of this polynomial

$\therefore$ $z^6 - 6z^5 + 15z^4 - 20z^3 + 15z^2 - 6z = z(z-2)(z^2 - z + 1)(z^2 - 3z + 3)$

(3 marks)