Question 10 (15 marks)

(a) (i) On the Argand diagram in Figure 8:

- mark and label a point to represent a complex number z
- hence mark and label the point representing $z+2$.

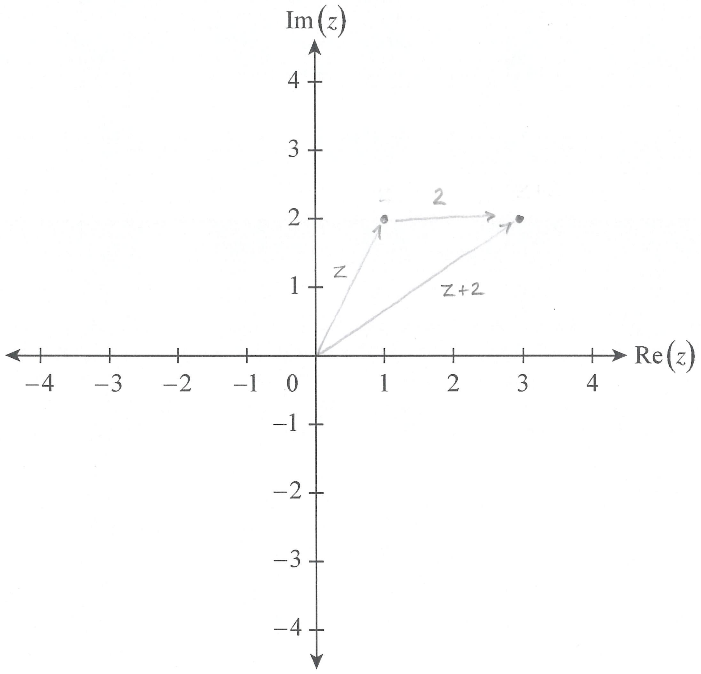


Figure 8 (1 mark)

(ii) Explain why $|z|+|z+2| \geq 2$.

By the triangle inequality, $|z| + |z+2| \geqslant |z+2-z|$

$= 2$

(2 marks)

(iii) Explain why the only solutions for $|z|+|z+2|=2$ are real.

Equality in the above occurs when 0, z, and z+2 are collinear

ie. when z and therefore z+2 are on the real axis

(1 mark)

(b) (i) On the Argand diagram in Figure 9:

(1) draw the set of all complex numbers z such that $|z+2|=|z|$. (2 marks)

(2) mark a point P, representing a complex number z such that $|z+2|=|z|$ and $\mathrm{Im}(z)>0$. (1 mark)

(3) mark the point Q, representing $z+2$. (1 mark)

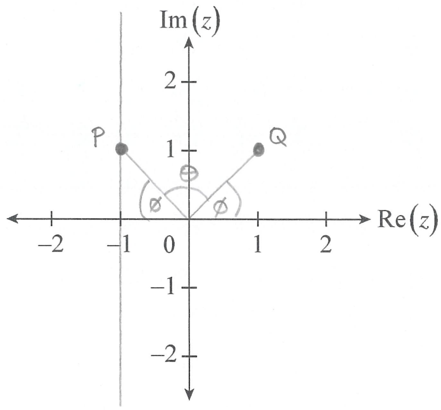


Figure 9

(ii) Let $\angle POQ=\theta$.

Show that $\dfrac{z}{z+2}=\operatorname{cis}\theta$.

Let $\arg(z) = \phi + \theta$

Then $\arg(z+2) = \phi$

$z = |z| \operatorname{cis}(\phi+\theta)$

$z+2 = |z| \operatorname{cis}(\phi)$ since $|z+2| = |z|$

$\therefore \dfrac{z}{z+2} = \dfrac{\cancel{|z|} \operatorname{cis}(\phi+\theta)}{\cancel{|z|} \operatorname{cis}(\phi)}$

$= \operatorname{cis}\theta$

(2 marks)

Question 10 continues on page 10.

(c) (i) Using z from part (b), and $\theta = \frac{\pi}{6}$, write $\frac{z}{z+2}$ in Cartesian form.

$\frac{z}{z+2} = \text{cis}\,\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i.\frac{1}{2}$

(1 mark)

(ii) Hence show that $z = -1 + \left(2+\sqrt{3}\right)i$.

From (b)(i), $z = -1 + ki$ and $z+2 = 1 + ki$ for some $k \in \mathbb{R}$

$\therefore \frac{-1+ki}{1+ki} = \frac{\sqrt{3}+i}{2}$

$-2+2ki = (\sqrt{3}+i)(1+ki)$

$= (\sqrt{3}-k) + i(\sqrt{3}k+1)$

Comparing real coefficients $\Rightarrow$ $-2 = \sqrt{3} - k$

$k = \sqrt{3}+2$

$\therefore z = -1 + (\sqrt{3}+2)i$

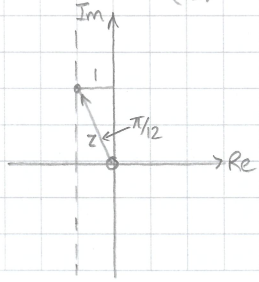
(2 marks)

(iii) Using Figure 9 or otherwise, show that z may be written in polar form as $z = \operatorname{cosec}\frac{\pi}{12}\,\text{cis}\,\frac{7\pi}{12}$.

From figure 9, $\arg(z+2) = \frac{1}{2}(\pi - \pi/6) = \frac{5\pi}{12}$

$\therefore \arg z = \frac{5\pi}{12} + \frac{\pi}{6} = \frac{7\pi}{12}$

$\therefore z = |z|\,\text{cis}\left(\frac{7\pi}{12}\right)$



But $\sin\left(\frac{\pi}{12}\right) = \frac{1}{|z|}$

$\therefore |z| = \operatorname{cosec}\left(\frac{\pi}{12}\right)$

$\therefore z = \operatorname{cosec}\left(\frac{\pi}{12}\right).\,\text{cis}\left(\frac{7\pi}{12}\right)$

(2 marks)