## Question 14 (15 marks)

The complex number $w$ is shown on the Argand diagram in Figure 14.

The dashed rays represent the complex numbers such that $\arg z = \frac{\pi}{6}$, $\arg z = \frac{\pi}{4}$, and $\arg z = \frac{\pi}{3}$.

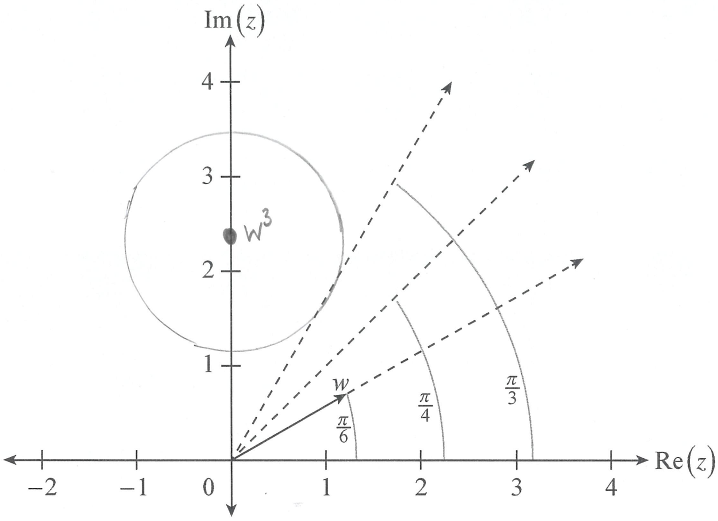


**Figure 14**

(a) On the Argand diagram in Figure 14:

(i) draw the approximate position of $w^3$. (1 mark)

(ii) sketch the set of complex numbers $z$ such that $|z - w^3| = |w|$. (2 marks)

(b) (i) Write $z = \frac{\sqrt{6}}{2} + \frac{i\sqrt{2}}{2}$ in exact polar form.

$z = \sqrt{2}\,\text{cis}\left(\frac{\pi}{6}\right)$

(1 mark)

(ii) Use de Moivre's theorem to show that $z^6 = -8$.

$z^6 = (2^{1/2})^6 . \text{cis}\left(6 \times \frac{\pi}{6}\right)$

$= 2^3 . \text{cis}\,\pi$

$= -8$

(1 mark)

(c) (i) Show that the solutions of $z^6 = -8$ are: $z_1 = \sqrt{2}\text{cis}\left(\frac{\pi}{6}\right)$; $z_2 = \sqrt{2}\text{cis}\left(\frac{\pi}{2}\right)$; $z_3 = \sqrt{2}\text{cis}\left(\frac{5\pi}{6}\right)$;

$z_4 = \sqrt{2}\text{cis}\left(-\frac{5\pi}{6}\right)$; $z_5 = \sqrt{2}\text{cis}\left(-\frac{\pi}{2}\right)$; and $z_6 = \sqrt{2}\text{cis}\left(-\frac{\pi}{6}\right)$.

$z^6 = -8$

$\therefore z^6 = 2^3 \text{cis}(\pi + k.2\pi)$ where $k \in \mathbb{Z}$

$\therefore z = 2^{1/2} \text{cis}\left(\frac{\pi}{6} + k.\frac{2\pi}{6}\right)$

ie. $z_1 = \sqrt{2}\,\text{cis}\left(\frac{\pi}{6}\right)$, $z_2 = \sqrt{2}\,\text{cis}\left(\frac{3\pi}{6}\right) = \sqrt{2}\,\text{cis}\left(\frac{\pi}{2}\right)$, $z_3 = \sqrt{2}\,\text{cis}\left(\frac{5\pi}{6}\right)$,

$z_4 = \sqrt{2}\,\text{cis}\left(\frac{-5\pi}{6}\right)$, $z_5 = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{6}\right) = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{2}\right)$, $z_6 = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{6}\right)$

(2 marks)

On the Argand diagram in Figure 15, $P$ represents $z_1^{\,3}$ and $Q$ represents $2z_1$, where $z_1 = \sqrt{2}\text{cis}\left(\frac{\pi}{6}\right)$.

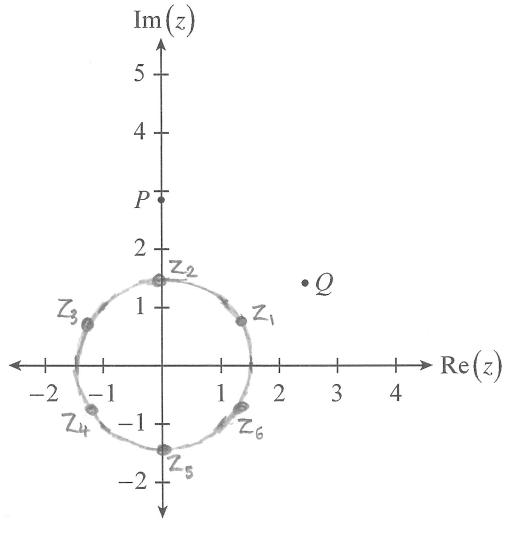


**Figure 15**

(ii) On the Argand diagram in Figure 15:

(1) sketch $|z| = \sqrt{2}$. (1 mark)

(2) draw and label $z_1, z_2, z_3, z_4, z_5$, and $z_6$. (2 marks)

(iii) Find the exact value of $\left|z_1^{\,3} - 2z_1\right|$.

$$|z_1^3 - 2z_1| = \left|2\sqrt{2}\,i - 2\sqrt{2}\left(\tfrac{\sqrt{3}}{2} + i.\tfrac{1}{2}\right)\right|$$
$$= |2\sqrt{2}\,i - \sqrt{6} - \sqrt{2}\,i|$$
$$= |-\sqrt{6} + \sqrt{2}\,i| = \sqrt{8} = 2\sqrt{2}$$

(2 marks)

Circles of radius $\sqrt{2}$ are drawn with centres $P$ and $Q$.

(iv) Deduce that these circles touch.

$|z_1^3 - 2z_1|$ gives the distance between P and Q, $2\sqrt{2}$ units.

This is exactly twice the radius of the circles centred at P and Q

∴ These circles touch

(1 mark)

(v) Find the point where the circles touch, and write it in exact $x + iy$ form.

The point where the circles touch is the midpoint of P and Q

$$= \frac{2\sqrt{2}\,i + \sqrt{6} + \sqrt{2}\,i}{2}$$

$$= \frac{\sqrt{6}}{2} + i.\frac{3\sqrt{2}}{2}$$

(2 marks)