

Question 10 (16 marks)

(a) (i) State the roots of the complex equation $w^6 = 1$ in $r \operatorname{cis} \theta$ form.

$z = \operatorname{cis}\left(0 + k \cdot \frac{2\pi}{6}\right)$ where $k \in \mathbb{Z}$
ie. $z_1 = \operatorname{cis}(0) = 1$, $z_2 = \operatorname{cis}\left(\frac{\pi}{3}\right)$, $z_3 = \operatorname{cis}\left(\frac{2\pi}{3}\right)$, $z_4 = \operatorname{cis}(\pi) = -1$, $z_5 = \operatorname{cis}\left(-\frac{2\pi}{3}\right)$, $z_6 = \operatorname{cis}\left(-\frac{\pi}{3}\right)$

(2 marks)

(ii) On the Argand diagram in Figure 10, plot the roots identified in part (a)(i).

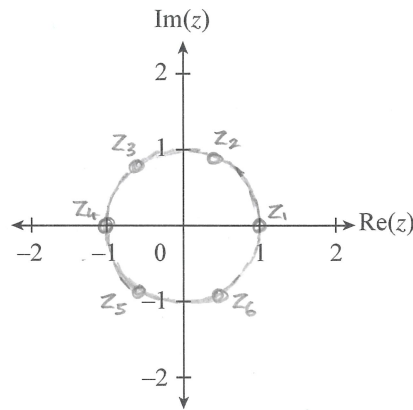


Figure 10

(2 marks)

(b) Consider the complex equation $(z-1)^6 = 1$.

(i) Using the roots identified in part (a)(i), state all the roots of the equation $(z-1)^6 = 1$, giving answers in any form.

$\operatorname{cis}(0)+1$, $\operatorname{cis}\left(\frac{\pi}{3}\right)+1$, $\operatorname{cis}\left(\frac{2\pi}{3}\right)+1$, $\operatorname{cis}(\pi)+1$, $\operatorname{cis}\left(-\frac{2\pi}{3}\right)+1$, $\operatorname{cis}\left(-\frac{\pi}{3}\right)+1$

(2 marks)

(ii) On the Argand diagram in Figure 11, plot the roots of the equation $(z-1)^6 = 1$.

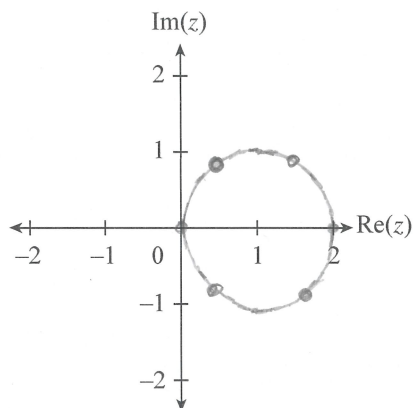


Figure 11

(1 mark)

(iii) Write the roots of $(z-1)^6 = 1$ in $r \operatorname{cis} \theta$ form or real form.

2,	$\frac{3}{2} + i\frac{\sqrt{3}}{2} = \sqrt{3} \operatorname{cis}\left(\frac{\pi}{6}\right)$,	$\frac{1}{2} + i\frac{\sqrt{3}}{2} = \operatorname{cis}\left(\frac{\pi}{3}\right)$
0,	$\frac{1}{2} - i\frac{\sqrt{3}}{2} = \operatorname{cis}\left(-\frac{\pi}{3}\right)$,	$\frac{3}{2} - i\frac{\sqrt{3}}{2} = \sqrt{3} \operatorname{cis}\left(-\frac{\pi}{6}\right)$

(2 marks)

(c) (i) Suppose that the polynomial $z^2 + bz + c$ has a zero $r \operatorname{cis} \theta$, where b and c are real, and $r > 0$ and $0 < \theta < \pi$.

Show that $b = -2r \cos \theta$ and $c = r^2$.

If $r \operatorname{cis} \theta = r(\cos \theta + i \sin \theta)$ is a zero of the polynomial
then $r \operatorname{cis}(-\theta) = r(\cos \theta - i \sin \theta)$ is also a zero of the polynomial
The sum of the roots is $2r \cos \theta$
The product of the roots is $r^2 (\cos^2 \theta + \sin^2 \theta) = r^2$
 \therefore The polynomial is $z^2 - 2r \cos \theta z + r^2$
ie. $b = -2r \cos \theta$ and $c = r^2$

(2 marks)

(ii) Verify that $(z-1)^6 = z^6 - 6z^5 + 15z^4 - 20z^3 + 15z^2 - 6z + 1$.

$$\begin{aligned}
 (z-1)^6 &= (z-1)^3 (z-1)^3 \\
 &= (z^3 - 3z^2 + 3z - 1)(z^3 - 3z^2 + 3z - 1) \\
 &= z^6 - 3z^5 + 3z^4 - z^3 - 3z^5 + 9z^4 - 9z^3 + 3z^2 + 3z^4 - 9z^3 + 9z^2 - 3z - z^3 + 3z^2 - 3z + 1 \\
 &= z^6 - 6z^5 + 15z^4 - 20z^3 + 15z^2 - 6z + 1
 \end{aligned}$$

(2 marks)

(d) Using part (b)(iii) and part (c), factorise $z^6 - 6z^5 + 15z^4 - 20z^3 + 15z^2 - 6z + 1$ into the product of real linear and real quadratic factors.

$$\begin{aligned}
 (z-1)^6 = 1 &\Rightarrow z^6 - 6z^5 + 15z^4 - 20z^3 + 15z^2 - 6z + 1 = 1 \\
 &\text{ie. } z^6 - 6z^5 + 15z^4 - 20z^3 + 15z^2 - 6z = 0 \\
 &\text{From (b)(iii), } \sqrt{3} \operatorname{cis}\left(\frac{\pi}{6}\right) \text{ is a zero of this polynomial} \\
 \therefore z^2 - 2\sqrt{3} \cos\left(\frac{\pi}{6}\right)z + 3 &= z^2 - 3z + 3 \text{ is a factor of this polynomial} \\
 &\text{Also from (b)(iii) } \operatorname{cis}\left(\frac{2\pi}{3}\right) \text{ is a zero of this polynomial} \\
 \therefore z^2 - 2\cos\left(\frac{2\pi}{3}\right)z + 1 &= z^2 - z + 1 \text{ is a zero of this polynomial} \\
 &\text{Also from (b)(iii) } 0 \text{ and } 2 \text{ are zeros of this polynomial} \\
 \therefore z^6 - 6z^5 + 15z^4 - 20z^3 + 15z^2 - 6z &= z(z-2)(z^2 - z + 1)(z^2 - 3z + 3)
 \end{aligned}$$

(3 marks)