QUESTION 13 (15 marks)

(a) (i) Solve $z^5 = -1$. Write your solutions in polar form.

$z^5 = \text{cis}(\pi + k.2\pi)$ where $k \in \mathbb{Z}$

$\therefore z = \text{cis}\left(\frac{\pi}{5} + k.\frac{2\pi}{5}\right)$

ie. $z = \text{cis}\left(\frac{-3\pi}{5}\right), \text{cis}\left(\frac{-\pi}{5}\right), \text{cis}\left(\frac{\pi}{5}\right), \text{cis}\left(\frac{3\pi}{5}\right), \text{cis}(\pi) = -1$

(3 marks)

(ii) Draw the solutions on the Argand diagram in Figure 14, labelling each solution in an anticlockwise direction from z_1 to z_5, where z_1 is the solution with the smallest positive argument.

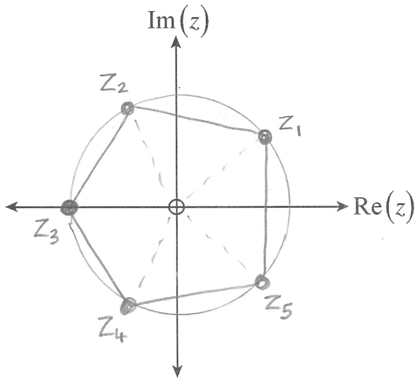

Figure 14

(2 marks)

Join your solutions labelled z_1, z_2, z_3, z_4, and z_5 to form a pentagon.

(iii) Show that $|z_1 - z_5| = 2\sin\frac{\pi}{5}$.

$|z_1 - z_5| = \left|\cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right) - \cos\left(\frac{-\pi}{5}\right) - i\sin\left(\frac{-\pi}{5}\right)\right|$

$= \left|\cancel{\cos\frac{\pi}{5}} + i\sin\frac{\pi}{5} - \cancel{\cos\frac{\pi}{5}} + i\sin\frac{\pi}{5}\right|$

$= 2\sin\frac{\pi}{5}$

(2 marks)

(iv) Show that the perimeter of the pentagon is $10\sin\frac{\pi}{5}$.

$|z_1 - z_5|$ is the length of one side of the regular pentagon

$\therefore$ Perimeter $= 10\sin\frac{\pi}{5}$

(1 mark)

(v) Show that the area of the pentagon is $\frac{5}{2}\sin\frac{2\pi}{5}$.

The pentagon consists of 5 isosceles triangles with equal side lengths 1 unit long and included angle $\frac{2\pi}{5}$

$\therefore$ Area $= 5 \times \frac{1}{2} \times 1 \times 1 \sin\frac{2\pi}{5}$

$= \frac{5}{2}\sin\frac{2\pi}{5}$

(2 marks)

Consider the solutions to $z^n = -1$ for integers $n \geq 3$.

(b) A polygon is obtained by plotting and joining the solutions of $z^n = -1$.

Let $P(n)$ be the perimeter of this polygon.

(i) Write down an expression for $P(n)$.

$P(n) = 2n\sin\frac{\pi}{n}$

(1 mark)

(ii) State the shape of the polygon formed as $n \to \infty$.

A circle!

(1 mark)

(iii) What exact value does $P(n)$ approach as $n \to \infty$?

2π

(1 mark)

(c) Let $A(n)$ be the area of the polygon described in part (b).

(i) Show that $A(n) = \frac{n}{2}\sin\frac{2\pi}{n}$.

The polygon consits n identical isosceles triangles with equal sides lengths 1 unit long and included angle $2\pi/n$

$\therefore A(n) = \frac{n}{2} \sin \frac{2\pi}{n}$

(1 mark)

(ii) What exact value does $A(n)$ approach as $n \to \infty$?

π

(1 mark)