Stage 2 Mathematical Methods

Discrete Random Variables Test

Topic 1: Subtopics 2.1, 2.2, 2.3

Total Marks – 40

<u>(Calculator and one A4 page of handwritten notes permitted)</u>

Question 1 (5 marks)

(a) (i) Each time a fair dice is rolled, $X = 1$ if the uppermost face is '3' and $X = 0$ otherwise.

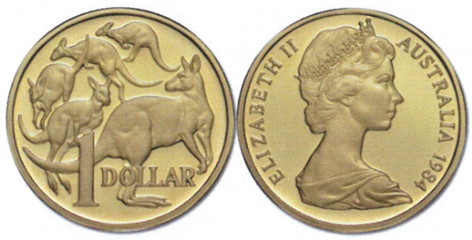
On the basis of the information above, is X a Bernoulli random variable? Tick the appropriate box.

Yes ☑

No ☐

(1 mark)

(ii) A fair coin is tossed five times. X is the number of times that 'heads' is uppermost.



On the basis of the information above, is X a Bernoulli random variable? Tick the appropriate box.

Yes ☐

No ☑

(1 mark)

(b) X represents a Bernoulli random variable with probability of success $p = 0.2$.

(i) State the mean of X.

$\mu_x = 0.2$

(1 mark)

(ii) Determine the standard deviation of X.

$\sigma_x = \sqrt{0.2 \times 0.8}$

$= 0.4$

(1 mark)

(c) X represents a Bernoulli random variable with probability of success p.

State the value of p for which the standard deviation of X is maximised.

$p = 0.5$

(1 mark)

Question 2 (8 marks)

(a) The discrete random variable X has the following probability distribution:

x	0	9	11
$P(X = x)$	a	$\frac{1}{10}$	$\frac{1}{15}$

(i) Find the value of a.

$a + \frac{1}{10} + \frac{1}{15} = 1$

$a = 1 - \frac{1}{10} - \frac{1}{15}$

$= \frac{5}{6}$ (0.833 3s.f.)

(1 mark)

(ii) Find the mean, μ_X.

$\mu_X = 0 \times \frac{5}{6} + 9 \times \frac{1}{10} + 11 \times \frac{1}{15}$

$= \frac{49}{30}$ (1.63 3s.f.)

(2 marks)

(iii) Find the standard deviation, σ_X.

$\sigma_X = \sqrt{0^2 \times \frac{5}{6} + 9^2 \times \frac{1}{10} + 11^2 \times \frac{1}{15} - \left(\frac{49}{30}\right)^2}$

$= 3.67$ (3s.f.)

(2 marks)

(b) People pay a fee to enter a local showground. Upon entry, each person receives 10 tokens that are redeemable for activities such as rides and games.

In one game, 30 balls numbered from 1 to 30 are placed into a bag. Players give two tokens to the game's operator each time they randomly draw one numbered ball from the bag. The number on the ball is checked and the ball is put back into the bag. The number on the ball determines the outcome of the game, as shown in the table below:

Number on the ball	*Numbers of tokens won from the operator by the player*	*Probability of occurring*
a multiple of 9	9	$\frac{1}{10}$
a multiple of 11	11	$\frac{1}{15}$

If the number on the ball drawn is neither a multiple of 9 nor a multiple of 11, the player wins no tokens from the operator.

(i) Let X represent the number of tokens won through playing this game.

What is the expected value of X?

$E(X) = \frac{49}{30}$ (1·63 3s.f.)

(1 mark)

(ii) In one day the game is played 1000 times.

Predict whether or not the operator will have more tokens at the end of the day than at the start of the day. Explain your answer.

For each game played, the operator receives 2 tokens

∴ For 1000 games, the operator receives 2000 tokens

For each game played, the expected number of tokens won from the operator is about 1·63.

∴ For 1000 games, the expected number of tokens won from the operator is about 1630.

∴ The operator would expect to have about 370 more tokens at the end of the day than at the start of the day.

(2 marks)

Question 3 (8 marks)

A seed wholesaler is selling a large quantity of seeds. From past experience, he knows that 80% of the seeds will germinate and grow into seedlings.

The manager of a plant nursery has purchased these seeds and is planning to germinate them in trays consisting of six pots (as shown in the photograph).

Only trays with one or more seedlings in all six pots can be sold.

The manager wants to make sure that more than 90% of trays can be sold.

(a) The manager decides to plant two seeds in every pot.

(i) Calculate the probability that, if two seeds are planted in a single pot, at least one of the seeds will germinate and grow into a seedling.

Let X = the number of seeds that germinate in a pot containing 2 seeds

$X \sim B(2, 0.80)$

$P(X \geqslant 1) = 0.96$

(2 marks)

(ii) Using your answer to part (i), calculate the probability that, if two seeds are planted in each of the six pots in each tray, at least one of the seeds in each pot will germinate and grow into a seedling.

Let Y = the number of pots that contain at least 1 seedling in a tray containing 6 pots

$Y \sim B(6, 0.96)$

$P(Y = 6) = 0.783$ (3s.f.)

(2 marks)

(b) If three seeds are planted in each of the six pots in each tray, is it likely that more than 90% of trays could be sold? Show calculations to support your answer.

Let X = the number of seeds that germinate in a pot containing 3 seeds

$X \sim B(3, 0.80)$

$P(X \geqslant 1) = 0.992$

Let Y = the number of pots that contain at least 1 seedling in a tray containing 6 pots

$Y \sim B(6, 0.992)$

$P(Y = 6) = 0.953$ (3s.f.) i.e. 95.3%

$\therefore$ It is more likely that more than 90% of trays can be sold.

(4 marks)

Question 4 (10 marks)

Consider a multiple-choice question that contains five possible answers from which to choose. Only one of the five answers is correct.

(a) A student attempts this multiple-choice question. Let $X = 1$ if the student randomly chooses the correct answer and $X = 0$ if the student randomly chooses an incorrect answer.

(i) Construct a table of the probability distribution of X.

x	0	1
$P(X=x)$	0.8	0.2

(2 marks)

(ii) Calculate the mean of X.

$\mu_X = 0.2$

(1 mark)

(b) A student attempts a test containing 50 of these multiple-choice questions. In order to pass the test, the student must correctly answer at least 25 of the questions.

(i) The student knows the correct answer to 15 questions, and chooses these answers. The student randomly chooses an answer for each of the remaining 35 questions.

(1) What is the probability that the student passes the test?

Let Y = the number of correct answers in the remaining 35 questions

$Y \sim B(35, 0.2)$

$P(Y \geq 10) = 0.146$ (3 s.f.)

(2 marks)

(2) In total, for this student, what is the expected number of correctly answered questions?

$E(Y+15) = 35 \times 0.2 + 15$

$= 22$

(2 marks)

(ii) To have a probability of at least 33% of passing the test, what is the minimum number of questions to which the student must know the correct answer, given that the student will randomly guess the answers to the remaining questions?

Suppose the student knows the correct answer to n of the 50 questions

Let Y = the number of correct answers in the remaining $50-n$ questions

$Y \sim B(50-n, 0.2)$

We require $P(Y \geq 25-n) \geq 0.33$

Using technology, $n \geq 17$

n	$P(Y \geq 25-n)$
15	0.1457
16	0.2269
17	0.3343

(3 marks)

Question 5 (9 marks)

In Australia, mobile (cell) phone numbers contain 10 digits, beginning with the digits 04. An example of a mobile phone number is 0402 199 999.

Mobile phone numbers are allocated to telecommunications service providers, for distribution to their customers. This allocation is made according to the table below.

Telecommunications service provider	*Allocated mobile phone numbers*
Optus ('Optus Mobile Pty Limited')	0401 xxx xxx 0402 xxx xxx 0403 xxx xxx
Vodafone ('Vodafone Australia Pty Limited')	0404 xxx xxx 0405 xxx xxx 0406 xxx xxx
Telstra ('Telstra Corporation Limited')	0407 xxx xxx 0408 xxx xxx 0409 xxx xxx 0400 xxx xxx

A survey company has developed a machine that calls, at random, mobile phone numbers that begin with '040'.

Assume that all mobile phone numbers that begin with 040 are valid, and that customers have not reallocated their mobile phone numbers to another telecommunications service provider.

(a) When the machine calls one mobile phone number:

(i) what is the probability that this mobile phone number is allocated to Telstra?

0.4

(1 mark)

(ii) what is the probability that this mobile phone number is ***not*** allocated to Telstra?

0.6

(1 mark)

(b) The machine calls batches of five mobile phone numbers.

Let X be the number of these calls that are made to mobile phone numbers that are allocated to Telstra.

(i) State the distribution of X.

$X \sim B(5, 0.4)$

(1 mark)

(ii) Within one batch of five mobile phone numbers:

(1) what is the probability that all five calls will be made to mobile phone numbers that are allocated to Telstra?

$P(X=5) = 0.0102$ (3s.f.)

(1 mark)

(2) what is the probability that at least one call will be made to a mobile phone number that is allocated to Telstra?

$P(X \geq 1) = 0.922$ (3s.f.)

(2 marks)

(c) How many mobile phone numbers would the machine need to call in order to have a minimum 99.5% chance of calling at least one mobile phone number that is allocated to Telstra?

Let X = the number of calls made to mobile phone numbers allocated to Telstra in a sample of n calls

$X \sim B(n, 0.4)$

We require $P(X \geq 1) \geq 0.995$

n	$P(X \geq 1)$
5	0.9222
⋮	⋮
10	0.9940
11	0.9964

Using technology, $n \geq 11$

(3 marks)