QUESTION 15 (15 marks)

(a) Consider the equation $x^3 + x^2 + x + 1 = 0$.

(i) Show that $x = -1$ is a root of this equation.

$(-1)^3 + (-1)^2 + (-1) + 1 = -1 + 1 - 1 + 1$

$= 0 \quad \therefore x = -1$ is a root of this equation

(1 mark)

(ii) Show that there are no other real roots.

$x^3 + x^2 + x + 1 = x^2(x+1) + 1(x+1)$

$= (x+1)(x^2+1)$

$= (x+1)(x+i)(x-i) \quad \therefore$ no real roots other than $x=-1$

(1 mark)

(b) Let $g(x) = \sqrt{x}$ and $h(x) = x^3 + x^2 + x + 1$.

(i) Find the composite function $g(h(x))$.

$g(h(x)) = \sqrt{x^3 + x^2 + x + 1}$

(1 mark)

(ii) State the domain of $g(h(x))$.

$x \geq -1$

(1 mark)

(c) Consider $f(x)=\sqrt{x^3+x^2+x+1}$.

(i) Draw the graph of $y=f(x)$ on the axes in Figure 17.

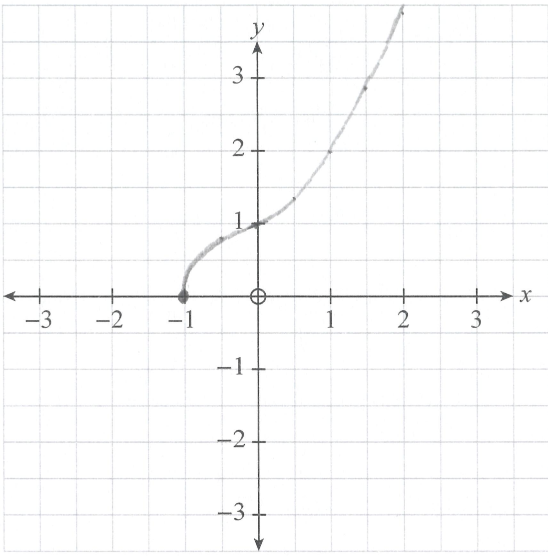

Figure 17

(2 marks)

(ii) Explain why $f(x)$ has an inverse, $f^{-1}(x)$.

$f(x)$ is a one-to-one function

$\therefore f(x)$ has an inverse, $f^{-1}(x)$

(1 mark)

(iii) Using your answer to part (c)(i), sketch the graph of $f^{-1}(x)$ on the axes in Figure 18.

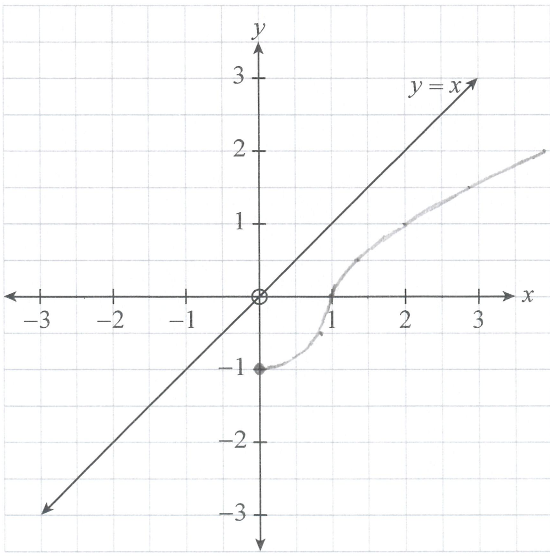

Figure 18

(2 marks)

(iv) (1) Find $f(1)$.

2

(1 mark)

(2) Find $f^{-1}(2)$.

1

(1 mark)

(d) If $y = f^{-1}(x)$, then $x = f(y)$.

(i) Use implicit differentiation to show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(f^{-1}(x)\right) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{f'(y)}$.

(1 mark)

(ii) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(f^{-1}(x)\right)$ at $x = 2$.

(3 marks)