**KEY FACTS AND CONCEPTS**

**Functions and sketching graphs**

* The **domain** of a function is the set of all possible *input* values for that function.
* The **range** of a function is the set of all possible *output* values for that function.
* If $f\left(x\right)$ and $g\left(x\right)$ are two functions, then the **composite function** $(f∘g)\left(x\right)=f\left(g\left(x\right)\right)$ if this exists.
* **One-to-one functions** satisfy both the vertical line test and the horizontal line test.
* **Many-to-one functions** satisfy the vertical line test but not the horizontal line test.
* If a function $f\left(x\right)$ is one-to-one, it *will* have an **inverse function** which we denote $f^{-1}\left(x\right)$.
* If a function $f\left(x\right)$ is many-to-one, it *will not* have an inversefunction.
* If $f\left(x\right)$ has an inverse function, then $f^{-1}\left(x\right)$:
	+ Has a graph which is the reflection of $y=f\left(x\right)$ in the line $y=x$.
	+ Satisfies $(f∘f^{-1})\left(x\right)=x$ and $ (f^{-1}∘f)\left(x\right)=x$.
* The **domain of** $f^{-1}\left(x\right)$is equal to the range of $f\left(x\right)$.
* The **range of** $f^{-1}\left(x\right)$ is equal to the domain of $f\left(x\right)$.
* Any function $f\left(x\right)$ which has an inverse, and whose graph is symmetrical about the line $y=x$ is a **self-inverse-function** with $f^{-1}\left(x\right)=f\left(x\right)$.
* A **reciprocal function** is a function of the form $f\left(x\right)=\frac{k}{x}$, where $k\ne 0$ is a constant.
* The **reciprocal of a function** $f\left(x\right)$ is $\frac{1}{f\left(x\right)}$.
* When $y=\frac{1}{f\left(x\right)}$ is graphed from $y=f\left(x\right)$:
	+ The zeros of $f\left(x\right)$ become vertical asymptotes of $\frac{1}{f\left(x\right)}$.
	+ The vertical asymptotes of $f\left(x\right)$ become zeros of $\frac{1}{f\left(x\right)}$.
	+ The local maxima of $f\left(x\right)$ become local minima of $\frac{1}{f\left(x\right)}$.
	+ The local minima of $f\left(x\right)$ become local maxima of $\frac{1}{f\left(x\right)}$.
	+ $f\left(x\right)>0⇔\frac{1}{f\left(x\right)}>0$.
	+ $f\left(x\right)<0⇔\frac{1}{f\left(x\right)}<0$.
	+ $f\left(x\right)\rightarrow 0^{\pm }⇔\frac{1}{f\left(x\right)}\rightarrow \pm \infty $.
	+ $f\left(x\right)\rightarrow \pm \infty ⇔\frac{1}{f\left(x\right)}\rightarrow 0^{\pm }$.
* A **rational function** is a function of the form $f\left(x\right)=\frac{p\left(x\right)}{q\left(x\right)}$ where $p\left(x\right)$ and $q\left(x\right)$ are polynomials.
* The **vertical asymptotes** of $f\left(x\right)=\frac{p\left(x\right)}{q\left(x\right)}$ correspond to the zeros of $q\left(x\right)$.
* The **non-vertical asymptotes** of $f\left(x\right)=\frac{p\left(x\right)}{q\left(x\right)}$ can be found by first using polynomial division and then considering the behaviour of $f\left(x\right)$ as $\rightarrow \pm \infty $.
* The **absolute value** or **modulus** of $x$ is $\left|x\right|=\left\{\begin{array}{c}x if x\geq 0\\ \\-x if x<0\end{array}\right.$
* The following are **properties of the absolute value**:
	+ $\left|x\right|\geq 0$ for all $x$.
	+ $\left|x\right|^{2}=x^{2}$ for all $x$.
	+ $\left|\frac{x}{y}\right|=\frac{\left|x\right|}{\left|y\right|}$ for all $x$ and $y$, $y\ne 0$.
	+ $\left|-x\right|=\left|x\right|$ for all $x$.
	+ $\left|xy\right|=\left|x\right|\left|y\right|$ for all $x$ and $y$.
	+ $\left|x-y\right|=\left|y-x\right|$ for all $x$ and $y$.
	+ $\left|x+y\right|\leq \left|x\right|+\left|y\right|$.
	+ $\left|x-y\right|\leq \left|x\right|+\left|y\right|$.
* To obtain the **graph of** $y=f\left(\left|x\right|\right)$ from the graph of $y=f\left(x\right)$:
	+ Discard the graph for $x<0$.
	+ Reflect the graph for $x\geq 0$ in the $y$-axis, keeping what was there.
	+ Points on the $y$-axis are invariant.
* To obtain the **graph of** $y=\left|f\left(x\right)\right|$ from the graph of $y=f\left(x\right)$:
	+ Keep the graph for $f\left(x\right)\geq 0$.
	+ Reflect the graph in the $x$-axis for $f\left(x\right)<0$, discarding what was there.
	+ Points on the $x$-axis are invariant.
* To **solve** **equations** **and** **inequalities** involving absolute value functions it is useful to remember that:
	+ If $\left|x\right|=a$ where $a>0$, then $x=\pm a$.
	+ If $\left|x\right|=\left|b\right|$ then $x=\pm b$.