Solutions

Stage 2 Specialist Mathematics

Functions and Sketching Graphs Test

Topic 3: Subtopics 3.1, 3.2, 3.3

Total Marks - 45

(Calculator not permitted. One A4 page of handwritten notes permitted.)

QUESTION 1 (9 marks)

Let $f(x) = \sqrt{x+3}$ and $g(x) = x^2 - 4x$.

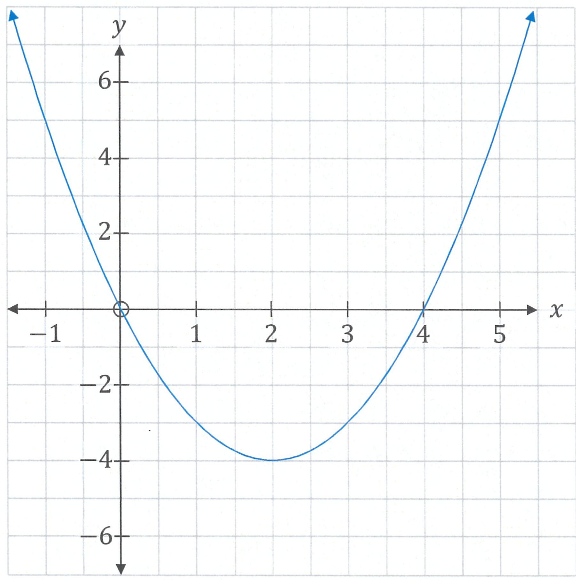
(a) State the domain of $f(x)$.

$x + 3 \geq 0$

$x \geq -3$

(1 mark)

(b) (i) Accurately draw the graph of $y = g(x)$ on the axes below:



(2 marks)

(ii) Hence state the range of $g(x)$.

$y \geq -4$

(1 mark)

(c) (i) Find the composite function $(f \circ g)(x)$.

$f \circ g(x) = f(x^2 - 4x)$

$= \sqrt{x^2 - 4x + 3}$

(1 mark)

(ii) Solve the equation $g(x) = -3$.

$x^2 - 4x = -3$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x = 1$ or $x = 3$

(2 marks)

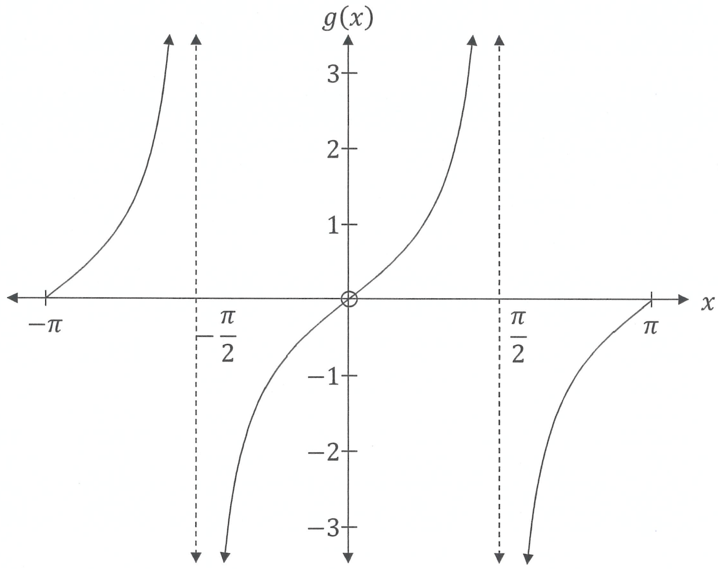
(iii) Hence, using your answer to (c) (ii) above and your graph from (b) (i), state the values of x for which $(f \circ g)(x)$ exists.

$x \leq 1 \cup x \geq 3$

(2 marks)

QUESTION 2 (9 marks)

The figure below shows the graph of $g(x) = 2\tan x$, where $-\pi \le x \le \pi$, $x \ne \pm\frac{\pi}{2}$.



(a) Explain why $g(x)$ is a function but does not have an inverse function.

$g(x)$ satisfies the vertical line test

$\therefore$ $g(x)$ is a function

$g(x)$ does not satisfy the horizontal line test

$\therefore$ $g(x)$ is not a one-to-one function

$\therefore$ $g(x)$ does not have an inverse function

(2 marks)

(b) (i) Explain why the following function *does* have an inverse function:

$$f(x) = 2\tan x, \text{ where } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

$f(x)$ satisfies the horizontal line test

$\therefore$ $f(x)$ is a one-to-one function

$\therefore$ $f(x)$ has an inverse function

(1 mark)

(ii) Show that $f^{-1}(x) = \arctan\frac{x}{2}$

$f(x)$: $y = 2\tan x$

$f^{-1}(x)$: $x = 2\tan y$

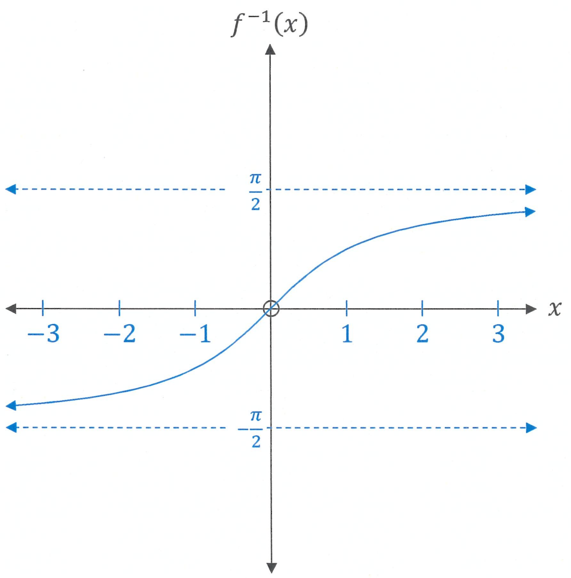
$\frac{x}{2} = \tan y$

$y = \arctan\frac{x}{2}$

$\therefore f^{-1}(x) = \arctan\frac{x}{2}$

(2 marks)

(iii) On the axes below, sketch $f^{-1}(x) = \arctan\frac{x}{2}$.



(2 marks)

(iv) State the domain and range of $f^{-1}(x)$ in exact form.

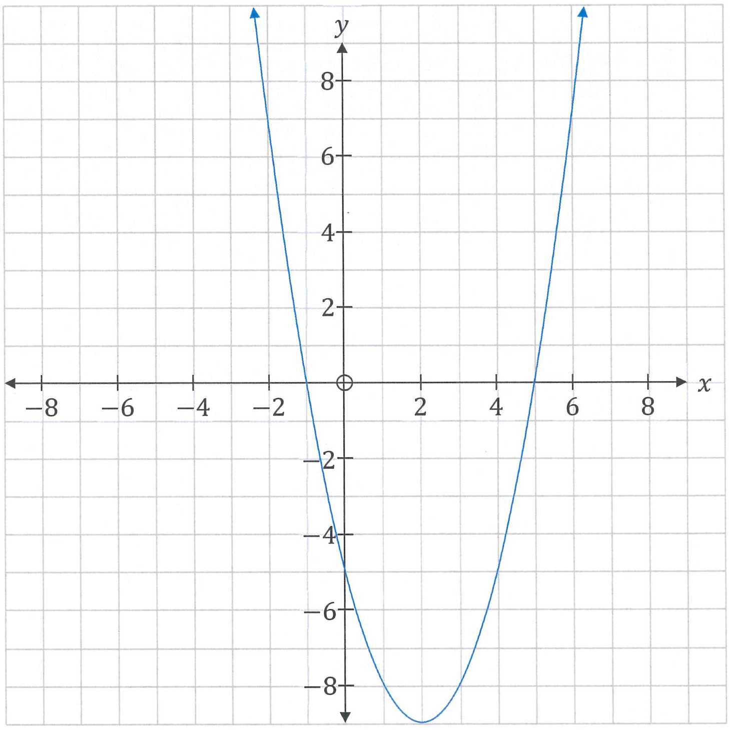
Domain : $x \in \mathbb{R}$

Range : $-\frac{\pi}{2} < y < \frac{\pi}{2}$

(2 marks)

Question 3 (9 marks)

(a) Accurately draw the graph of $y = (x+1)(x-5)$ on the axes below:

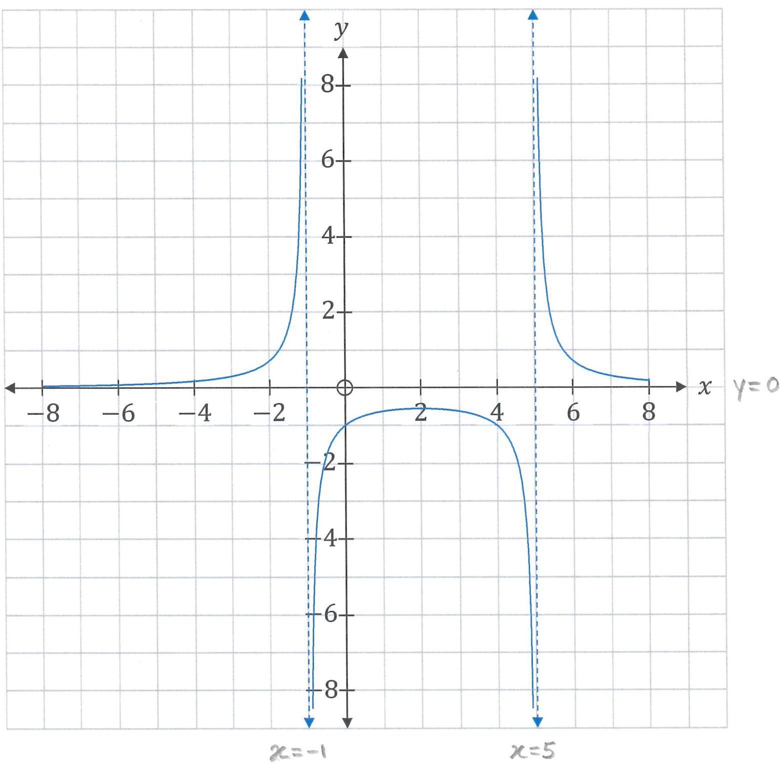


(2 marks)

(b) Let $f(x) = \frac{5}{(x+1)(x-5)}$.

(i) Draw the graph of $y = f(x)$ on the axes below:

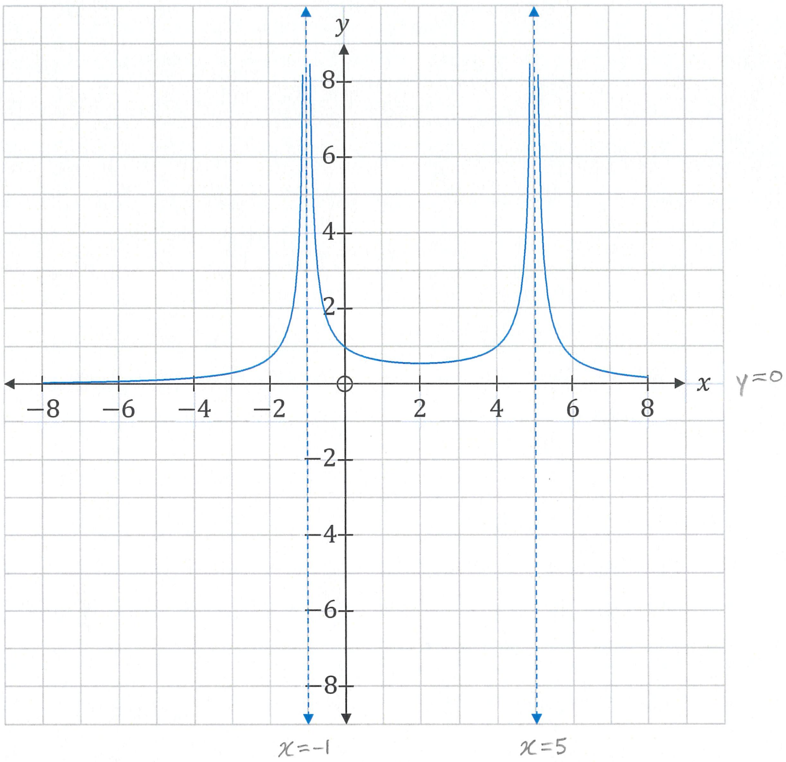
Clearly show the behaviour of the function near the asymptote(s).



(3 marks)

(ii) Draw the graph of $y = |f(x)|$ on the axes below:

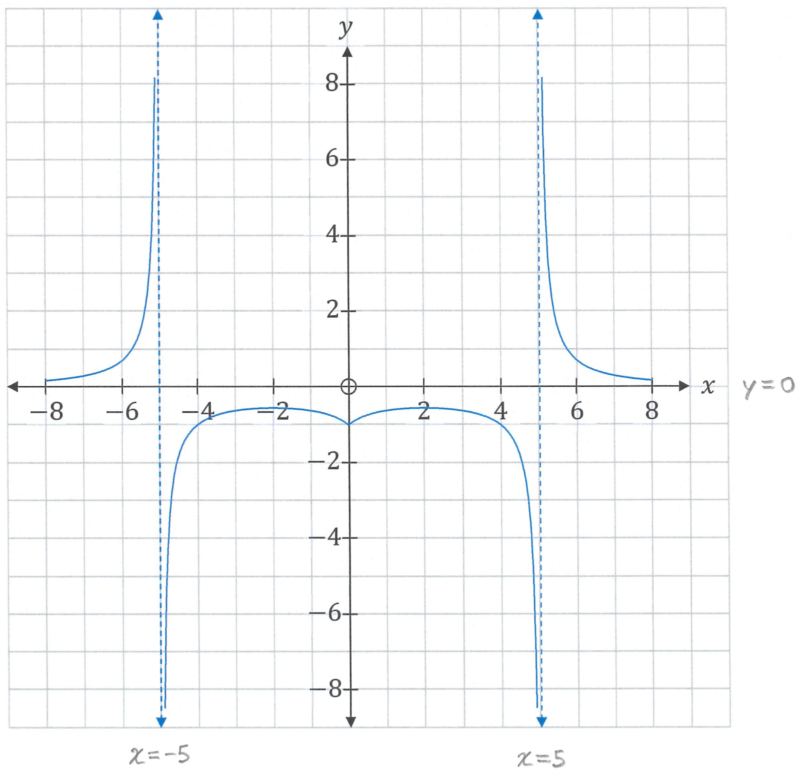
Clearly show the behaviour of the function near the asymptote(s).



(1 mark)

(iii) Draw the graph of $y = f(|x|)$ on the axes below:

Clearly show the behaviour of the function near the asymptote(s).



(3 marks)

Question 4 (9 marks)

Let $f(x) = \frac{x^2 - 8x + 16}{2x - 4}$. [Bonus information: $f'(x) = \frac{2x(x-4)}{(2x-4)^2}$]

(a) Use synthetic division to write $f(x)$ in the form $f(x) = q(x) + \frac{r(x)}{(x-\alpha)}$.

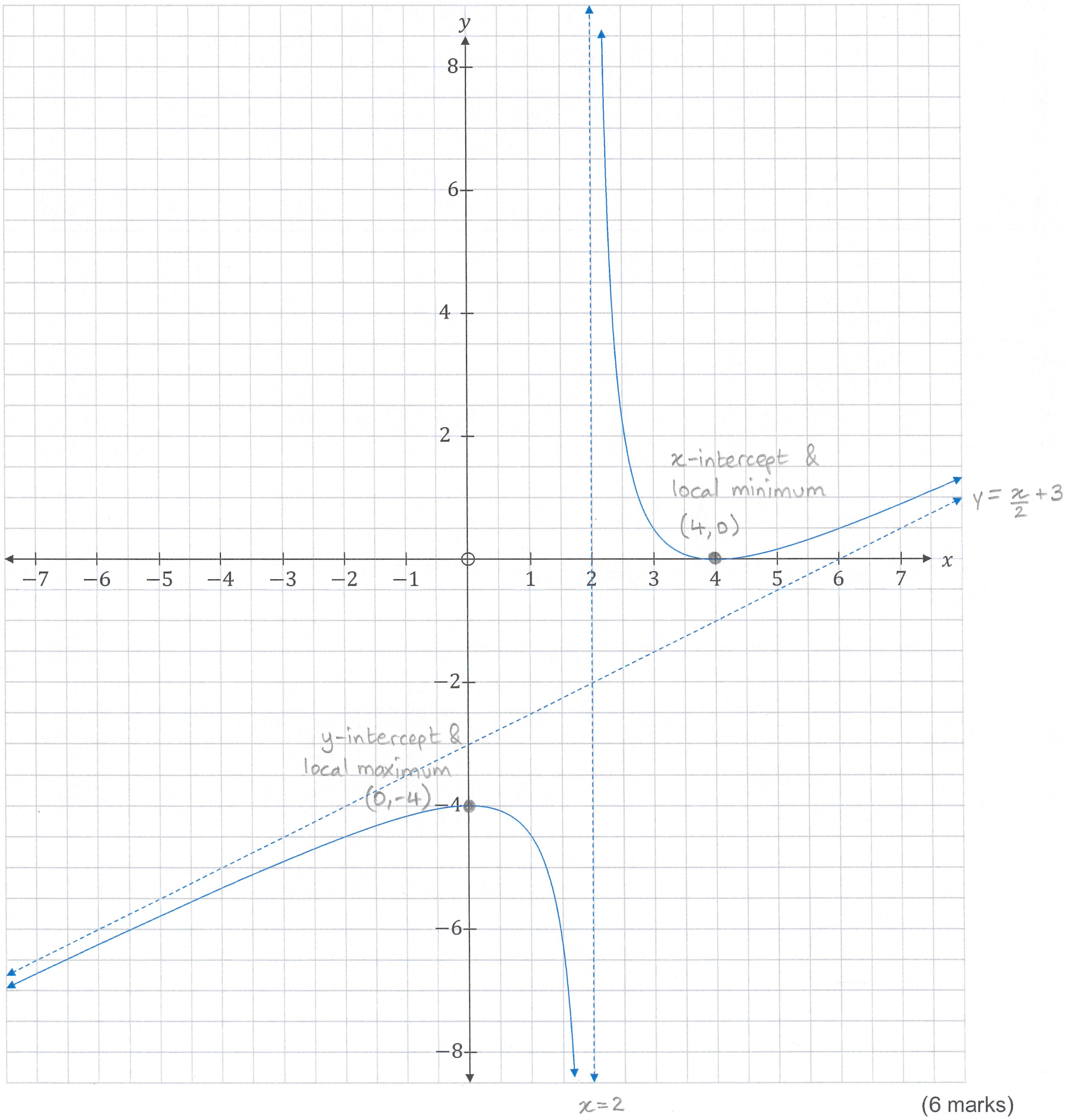
$$\begin{array}{c|ccc} 2 & \frac{1}{2} & -4 & 8 \\ & 0 & 1 & -6 \\ \hline & \frac{1}{2} & -3 & 2 \end{array}$$

$\therefore f(x) = \frac{x}{2} - 3 + \frac{2}{x-2}$

(3 marks)

(b) Hence sketch the graph of $y = f(x)$ on the axes below.

Clearly show the axes intercepts and the behaviour of the function near the asymptote(s).



(6 marks)

Question 5 (9 marks)

Let $f(x) = \frac{2x^2}{x^2+x-2}$. [Bonus information: $f'(x) = \frac{2x(x-4)}{(x^2+x-2)^2}$]

(a) Use polynomial division and factorisation to write $f(x)$ in the form $f(x) = q(x) + \frac{r(x)}{(x-\alpha)(x-\beta)}$.

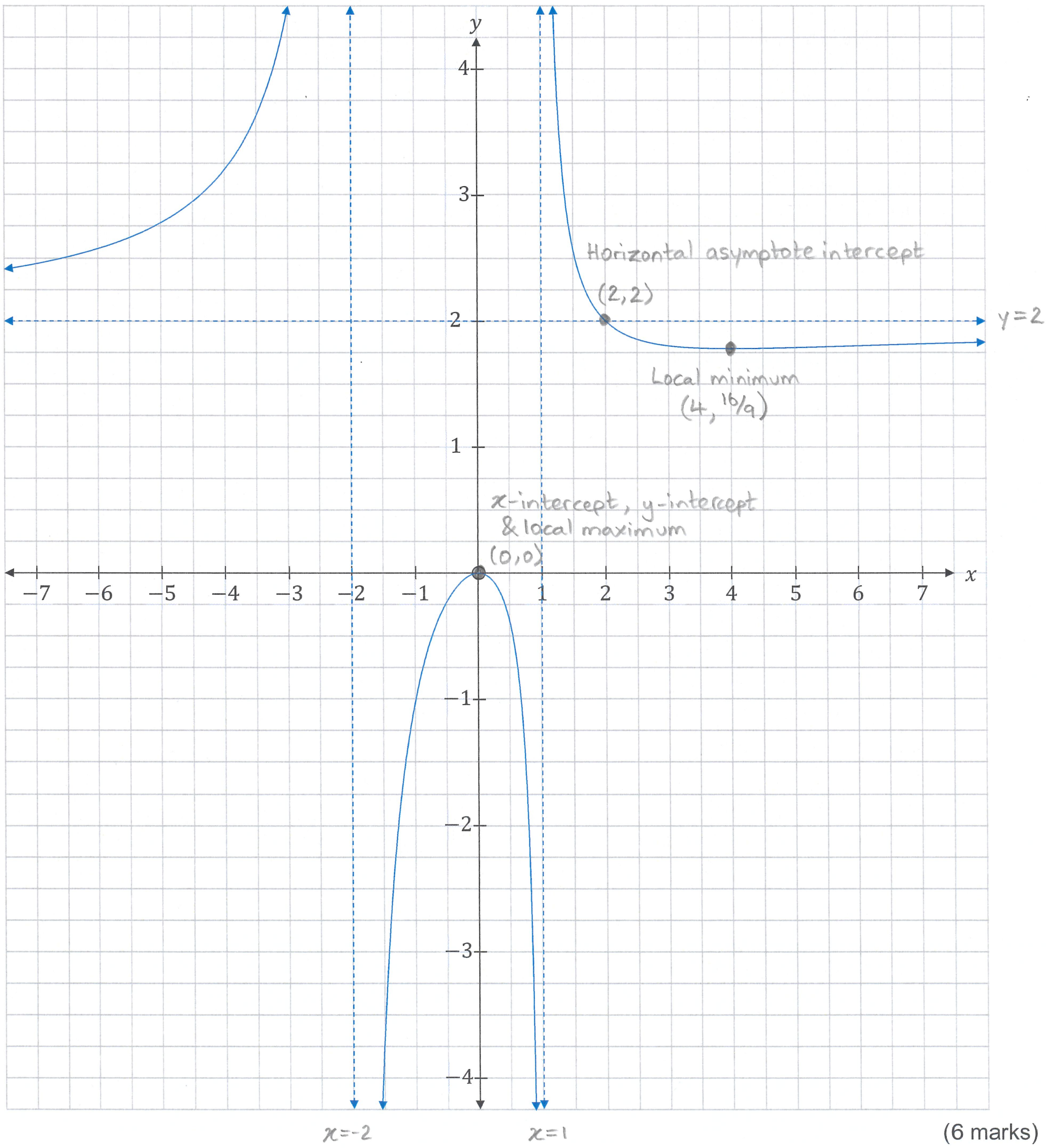
$$\begin{array}{r|l} & 2 \\ \hline x^2+x-2 & 2x^2+0x+0 \\ & 2x^2+2x-4 \\ \hline & -2x+4 \end{array}$$

$\therefore f(x) = 2 - \dfrac{2x-4}{(x+2)(x-1)}$

(3 marks)

(b) Hence sketch the graph of $y = f(x)$ on the axes below.

Clearly show axes and other intercepts and the behaviour of the function near the asymptote(s).



(6 marks)