Question 10 (9 marks)

Some functions do not have antiderivatives that can be expressed in terms of standard mathematical functions. For these functions, only an estimate of the area between the graph of the function and the x-axis for a given domain can be found.

Consider one such function: $f(x) = x^x$ for $x > 0$.

Figure 10 shows the graph of $y = f(x)$ and four rectangles of equal width that can be used to calculate an underestimate of the area between the graph of $y = f(x)$ and the x-axis for $1 \leq x \leq 3$.

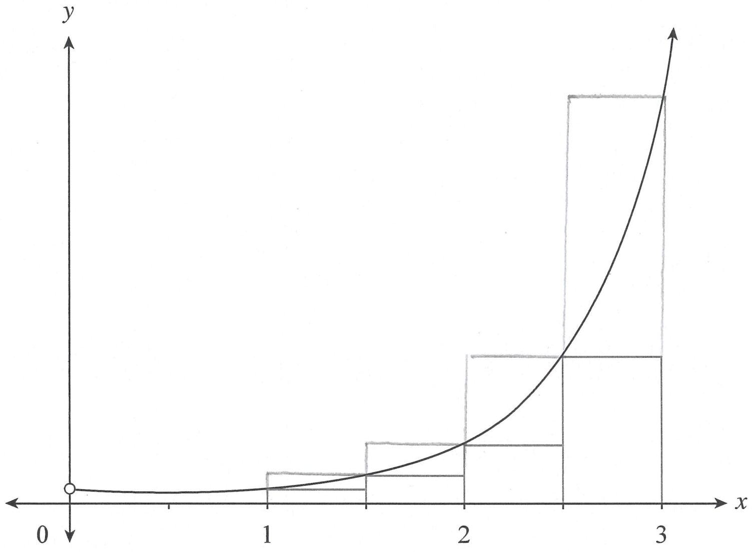


Figure 10

(a) Calculate this underestimate, expressing your answer correct to *five* significant figures.

Underestimate $= 0.5 \times f(1) + 0.5 \times f(1.5) + 0.5 \times f(2) + 0.5 \times f(2.5)$

$= 8.3596$ units^2

(2 marks)

(b) *On the set of axes in Figure 10 above*, add four rectangles of equal width that could be used to calculate an overestimate of the area between the graph of $y = f(x)$ and the x-axis for $1 \leq x \leq 3$.

(1 mark)

(c) (i) Determine $f''(x)$, given that the derivative of $f(x)=x^x$ for $x>0$ is $f'(x)=x^x(\ln x+1)$.

$$f''(x) = x^x.(\ln x+1)^2 + x^x.\frac{1}{x}$$

$$= x^x(\ln x+1)^2 + x^{x-1}$$

(2 marks)

(ii) Hence, show that the graph of $y=f(x)$ for $x>0$ is always convex (i.e. concave up).

Since $x>0$, $x^x>0$ and $x^{x-1}>0$

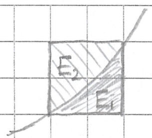
Also $(\ln x+1)^2 \geq 0$

$\therefore f''(x)>0$

$\therefore f(x)$ is always concave upwards

(2 marks)

(iii) Hence, when using rectangles of equal width to approximate the area bounded by the graph of $y=f(x)$ and the x-axis for $1\le x\le 3$, is it more accurate to use an underestimate or overestimate? Justify your answer.



E_1 = error from underestimate

E_2 = error from overestimate

$E_1 < E_2$ for a curve which is concave upwards

$\therefore$ An underestimate is more accurate

(2 marks)