Question 14 (12 marks)

Consider the curve defined by $y = f_k(x)$ where

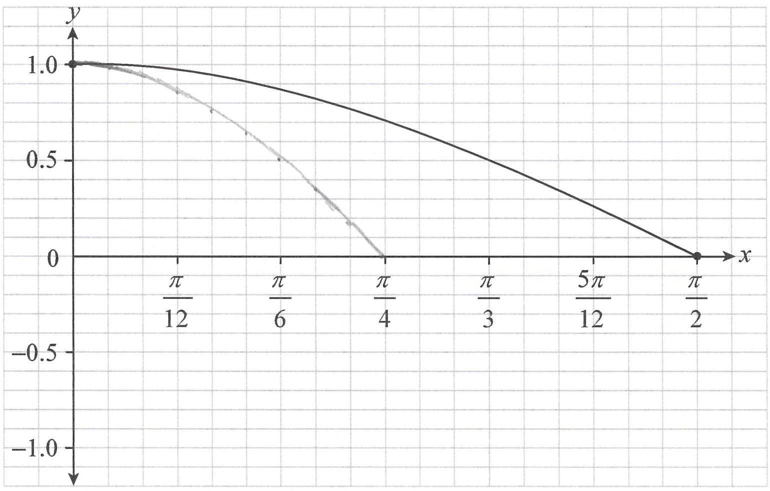
$$f_k(x) = \cos kx$$

and k is a positive integer.

The domain of the function $f_k(x)$ is dependent on the value of k, such that $0 \le x \le \frac{\pi}{2k}$.

(a) If $k = 1$ for $y = f_k(x)$ then the function is $f_1(x) = \cos 1x$.

The corresponding domain for the function of $f_1(x)$ is therefore $0 \le x \le \frac{\pi}{2}$. The graph of $f_1(x)$ is shown below.



Using algebra, calculate the area between $y = f_1(x)$ and the x-axis over the domain of the function.

$$\text{Area} = \int_0^{\pi/2} \cos x \, dx$$

$$= \Big[\sin x \Big]_0^{\pi/2}$$

$$= \sin \pi/2 - \cancel{\sin 0}^{\,0}$$

$$= 1 \text{ units}^2$$

(3 marks)

(b) Let $k = 2$ for $y = f_k(x)$.

(i) State $f_2(x)$ as a function.

$f_2(x) = \cos 2x$

(1 mark)

(ii) State the domain of the function $f_2(x)$.

$0 \leqslant x \leqslant \pi/4$

(1 mark)

(iii) On the axes on page 12, sketch the curve of $y = f_2(x)$ over the domain of the function $f_2(x)$.

(1 mark)

(iv) Calculate the area between $y = f_2(x)$ and the x-axis over the domain of the function.

Using technology, $A = \int_0^{\pi/4} \cos 2x \, dx$

$= \frac{1}{2}$ units^2

(1 mark)

(c) Find the value of $\int_0^{\frac{\pi}{6}} \cos 3x \, \mathrm{d}x$.

$\frac{1}{3}$

(1 mark)

(d) (i) Make a conjecture about the exact value of the area bounded by $y = f_k(x)$ and the x-axis over $0 \le x \le \dfrac{\pi}{2k}$, for any value of k.

Area $= \dfrac{1}{k}$ units^2

(1 mark)

(ii) Prove or disprove your conjecture.

Area $= \displaystyle\int_0^{\pi/2k} \cos kx \, dx$

$= \left[\dfrac{1}{k} \sin kx \right]_0^{\pi/2k}$

$= \left(\dfrac{1}{k} \cdot \sin \dfrac{\pi}{2} \right) - \cancel{\left(\dfrac{1}{k} \sin 0 \right)}^{0}$

$= \dfrac{1}{k}$ [proved]

(3 marks)