Question 15 (14 marks)

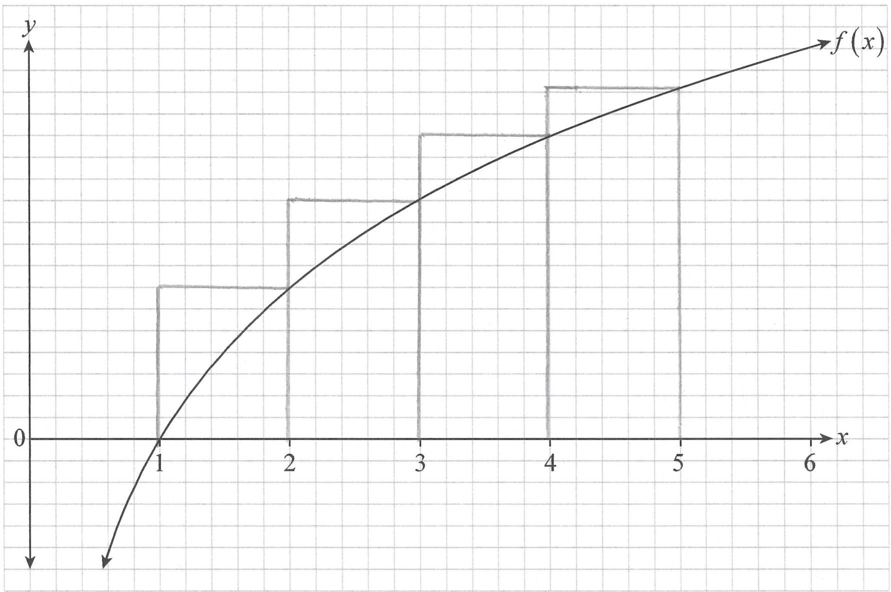
(a) Show that, if $y = x\ln x - x$, then $\dfrac{dy}{dx} = \ln x$.

$$\frac{dy}{dx} = 1.\ln x + \cancel{x}.\frac{1}{\cancel{x}} - 1$$

$$= \ln x$$

(2 marks)

Consider the function $f(x) = \ln x$. The graph of $y = f(x)$ is shown below for $x > 0$.



(b) An overestimate of the area between the graph of $y = f(x)$ and the x-axis from $x = 1$ to $x = 5$ is to be calculated, using four rectangles of equal width.

(i) On the graph above, draw the four rectangles used to determine this overestimate.

(1 mark)

(ii) Calculate this overestimate, giving your answer as an *exact* value.

Overestimate $= 1\times f(2) + 1\times f(3) + 1\times f(4) + 1\times f(5)$

$= \ln 2 + \ln 3 + \ln 4 + \ln 5$

$= \ln 120$ units^2

(2 marks)

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(c) Consider the definition that $n! = n\times(n-1)\times(n-2)\times\ldots\times 2\times 1$.

Explain why

$$\ln n! = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n$$
$$= \ln 2 + \ln 3 + \cdots + \ln n.$$

$\ln n! = \ln n + \ln(n-1) + \ldots + \ln 3 + \ln 2 + \ln 1$ [using $\ln(AB) = \ln A + \ln B$ repeatedly]

Also $\ln 1 = 0$ $\therefore \ln n! = \ln 2 + \ln 3 + \ldots + \ln n$

(1 mark)

(d) (i) Find an expression for an overestimate of the area between the graph of $y = f(x)$ and the x-axis from $x = 1$ to $x = n$, using n rectangles of equal width.

Overestimate $= \ln 2 + \ln 3 + \ldots + \ln n$

$= \ln n!$

(1 mark)

(ii) Hence, explain why $\ln n! > \int_1^n \ln x \, dx$.

$\int_1^n \ln x \, dx$ represents the exact area under the curve from $x=1$ to $x=n$

$\ln n!$ is an overestimate of this same area

$\therefore \ln n! > \int_1^n \ln x \, dx$

(2 marks)

(iii) Using the information in part (a), evaluate $\int_1^n \ln x \, dx$.

$\int_1^n \ln x \, dx = \left[x \ln x - x\right]_1^n$ using part (a) above

$= (n.\ln n - n) - (1.\cancel{\ln 1}^{0} - 1)$

$= n.\ln n + 1 - n$

(2 marks)

(iv) Hence, use the inequality given in part (d)(ii) and your answer to part (d)(iii) to show that

$$n! > n^n \times e^{1-n}.$$

$\ln n! > n.\ln n + 1-n$

$= \ln n^n + \ln e^{1-n}$

$= \ln(n^n . e^{1-n})$

$\therefore n! > n^n . e^{1-n}$, since $\ln A > \ln B \Leftrightarrow A > B$

(3 marks)