Question 7 (14 marks)

(a) (i) Find $\frac{dy}{dx}$ if $y = xe^{-x}$.

$\frac{dy}{dx} = 1.e^{-x} - x.e^{-x}$

(2 marks)

(ii) Hence show that $\int xe^{-x}dx = -xe^{-x} - e^{-x} + c$.

From (i) above, $\int e^{-x} - x.e^{-x}\,dx = xe^{-x} + c$

$\int e^{-x}dx - \int x.e^{-x}dx = xe^{-x} + c$

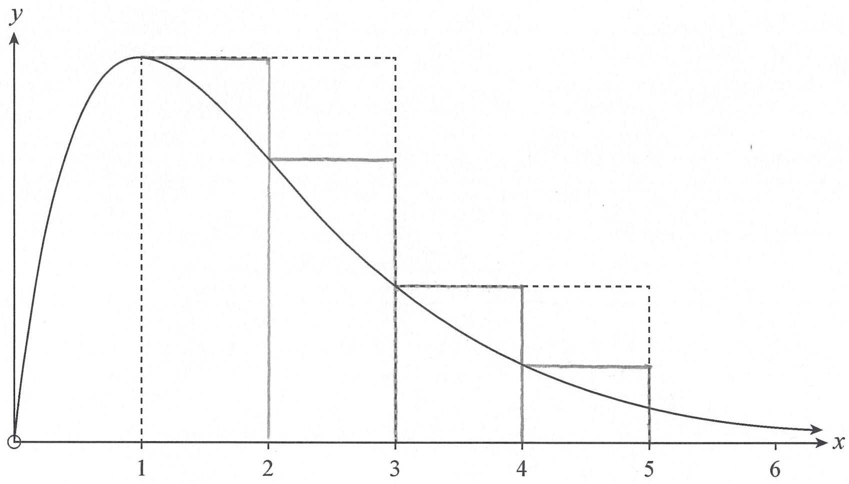
$-e^{-x} - \int x.e^{-x}dx = x.e^{-x} + c$

$-\int x.e^{-x}dx = x.e^{-x} + e^{-x} + c$

$\int x.e^{-x}dx = -xe^{-x} - e^{-x} + c$

(3 marks)

The graph of $y = f(x)$, where $f(x) = xe^{-x}$, is shown below.



(b) An estimate is required for the area bounded by $f(x)$, the x-axis, and the vertical lines $x = 1$ and $x = 5$.

(i) Two rectangles, each 2 units wide, have been added to the graph to be used in the calculation of an overestimate for this area.

Calculate this overestimate by finding the sum (S) of the areas of these two rectangles, correct to three decimal places.

$S = 2 \times f(1) + 2 \times f(3)$

$= 1.034 \text{ units}^2$

(2 marks)

(ii) A new overestimate of the same area can be calculated using *four* rectangles of equal width.

(1) On the graph above, draw four rectangles that could be used to produce a new overestimate. (1 mark)

(2) Calculate the new overestimate, giving your answer correct to three decimal places.

$S = 1 \times f(1) + 1 \times f(2) + 1 \times f(3) + 1 \times f(4)$

$= 0.861 \text{ units}^2$

(2 marks)

(c) With reference to part (a)(ii), find the exact value of the area bounded by $f(x)$, the x-axis, and the vertical lines $x=1$ and $x=5$.

$$\text{Exact area} = \int_1^5 x.e^{-x}\,dx$$

$$= \left[-x.e^{-x} - e^{-x}\right]_1^5$$

$$= \left(-5.e^{-5} - e^{-5}\right) - \left(-1.e^{-1} - e^{-1}\right)$$

$$= 2e^{-1} - 6e^{-5} \text{ units}^2$$

(3 marks)

(d) Compare your overestimate calculations from part (b) with your answer to part (c).

Comment on the effect that increasing the number of rectangles used in your calculations has on the accuracy of the estimates obtained.

Increasing the number of rectangles improves the accuracy of the estimate

(1 mark)