Name/Registration Number: ....

Stage 2 Specialist Mathematics – Skills and Applications Task

This Skills and Applications Task is divided into two parts. Part 1 is to be completed without a calculator or notes. For Part 2, you may have access to your graphics calculator and one A4 page of handwritten notes. You will commence with both parts of the task but will not have access to your calculator or notes until Part 1 is collected.

Topic 4: Vectors in Three Dimensions

PART 1: NO CACULATOR or NOTES **20 minutes** **Total: 16 marks**

QUESTION 1 (5 marks)

Find k such that the angle between the vectors $\underset{\sim}{a} = [k\ , 1, 1]$ and $\underset{\sim}{b} = [1, k, 1]$ is $\frac{\pi}{3}$.

$\cos\frac{\pi}{3} = \frac{\underline{a} \bullet \underline{b}}{|\underline{a}|.|\underline{b}|}$

$\frac{1}{2} = \frac{2k+1}{k^2+2}$

$k^2+2 = 4k+2$

$k^2-4k=0$

$k(k-4)=0$

$k=0$ or $k=4$

(5 marks)

QUESTION 2 (8 marks)

In the following system of equations, k is a real constant.

$$\begin{aligned} x - 2y + z &= 4 \\ -x + 4y + kz &= 0 \\ 2x - 2y - z &= k \end{aligned}$$

(a) Write the system in augmented matrix form and, stating all row operations used, show that this leads to the reduced system: $\begin{bmatrix} 1 & -2 & 1 & : & 4 \\ 0 & 2 & k+1 & : & 4 \\ 0 & 0 & k+4 & : & 12-k \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -1 & 4 & k & 0 \\ 2 & -2 & -1 & k \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 2 & k+1 & 4 \\ 0 & -2 & 3 & 8-k \end{array}\right] \begin{array}{l} \\ R_1 + R_2 \\ 2R_1 - R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 2 & k+1 & 4 \\ 0 & 0 & k+4 & 12-k \end{array}\right] \begin{array}{l} \\ \\ R_2 + R_3 \end{array}$$

(3 marks)

(b) State the value of k for which the system has no solution.

$k = -4$

(1 mark)

(c) (i) Consider the case where $k = 12$ and solve the system.

$$k=12 \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 2 & 13 & 4 \\ 0 & 0 & 16 & 0 \end{array}\right]$$

$R_3 \Rightarrow z=0$ $\quad$ $R_2 \Rightarrow 2y=4$ $\quad$ $R_1 \Rightarrow x-4=4$

$y=2$ $\quad$ $x=8$

(3 marks)

(ii) Give a geometrical interpretation of the system of equations for the value of k found used in part (c) (i).

Three planes intersecting at the point $(8, 2, 0)$

(1 mark)

QUESTION 3 (3 marks)

Using the formulae $\cos\theta = \dfrac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}$ and $\sin\theta = \dfrac{|\underset{\sim}{u}\times\underset{\sim}{v}|}{|\underset{\sim}{u}||\underset{\sim}{v}|}$ prove that if $\underset{\sim}{u}\cdot\underset{\sim}{v} = |\underset{\sim}{u}\times\underset{\sim}{v}|$ then the angle between $\underset{\sim}{u}$ and $\underset{\sim}{v}$ must be $45°$. The vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are non-zero vectors.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{|\underline{u}\times\underline{v}|}{\cancel{|\underline{u}|.|\underline{v}|}} \cdot \frac{\cancel{|\underline{u}|.|\underline{v}|}}{\underline{u}\cdot\underline{v}}$$

$$= 1 \quad \text{if } \underline{u}\cdot\underline{v} = |\underline{u}\times\underline{v}|$$

$$\therefore \theta = \tan^{-1}(1)$$

$$= 45°$$

(3 marks)

END OF PART 1

Part 1 will be collected at the end of 20 minutes. Be sure to check your work thoroughly before handing it up at the end of the allocated time.

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PART 2: CACULATOR AND NOTES MAY BE USED

TIME: 50 minutes **TOTAL: 36 marks**

QUESTION 4 (5 marks)

(a) Find the equation of the line passing through the point $(1, -2, 4)$ in the direction of the vector $[-1, 2, -3]$.

$\underline{r} = [1, -2, 4] + \lambda[-1, 2, -3]$

i.e. $\begin{cases} x = 1 - \lambda \\ y = -2 + 2\lambda \\ z = 4 - 3\lambda \end{cases}$

(2 marks)

(b) Find the coordinates of the point where the line intersects the plane

$$2x - y + 3z = -10.$$

Substituting the parametric equations of the line into the equation of the plane gives

$2(1-\lambda) - (-2+2\lambda) + 3(4-3\lambda) = -10$

$16 - 13\lambda = -10$

$-13\lambda = -26$

$\lambda = 2$

$\therefore$ the point of intersection is $(-1, 2, -2)$

(3 marks)

QUESTION 5 (11 marks)

(a) Points A (1, 3, 0), B (5, -1, 2) and C (-2, 5, -2) are three points on a plane P_1.

(i) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

$\overrightarrow{AB} = [4, -4, 2]$

$\overrightarrow{AC} = [-3, 2, -2]$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 4 & -4 & 2 \\ -3 & 2 & -2 \end{vmatrix} = [4, 2, -4]$$

(3 marks)

(ii) Find the area of triangle ABC.

$$\text{Area of } \Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= \frac{1}{2}\sqrt{4^2 + 2^2 + 4^2}$$

$$= 3 \text{ units}^2$$

(2 marks)

(iii) Find the equation of the plane P_1.

$\overrightarrow{AB} \times \overrightarrow{AC}$ is normal to plane P_1

$\therefore$ the equation of P_1 is $4x + 2y - 4z = 4(1) + 2(3) - 4(0)$

$4x + 2y - 4z = 10$

i.e. $2x + y - 2z = 5$

(2 marks)

(b) Plane P_2 is parallel to plane P_1 and passes through D (7, 3, -3).

(i) Find the equation of plane P_2.

Since plane P_2 is parallel to plane P_1, $\overrightarrow{AB} \times \overrightarrow{AC}$ is normal to plane P_2

$\therefore$ the equation of P_2 is $4x + 2y - 4z = 4(7) + 2(3) - 4(-3)$

$4x + 2y - 4z = 46$

i.e. $2x + y - 2z = 23$

(2 marks)

(i) Find the distance between the two planes.

The equation of plane P_1 is $2x + y - 2z - 5 = 0$ and $(7, 3, -3)$ is a point on P_2

$$\therefore d = \frac{|2(7) + (3) - 2(-3) - 5|}{\sqrt{2^2 + 1^2 + 2^2}}$$

$= 6$ units

(2 marks)

QUESTION 6 (15 marks)

Let A (3, 4, 0), B (-1, 6, 4), C (-5, 2, 2) and D (-1, 0, -2) be four points in space.

(a) Show that A, B, C and D lie in the plane $x - 2y + 2z = -5$.

A: $(3) - 2(4) + 2(0) = -5$ $\therefore$ A lies in the plane

B: $(-1) - 2(6) + 2(4) = -5$ $\therefore$ B lies in the plane

C: $(-5) - 2(2) + 2(2) = -5$ $\therefore$ C lies in the plane

D: $(-1) - 2(0) + 2(-2) = -5$ $\therefore$ D lies in the plane

(4 marks)

(b) Show that A, B, C and D form the vertices of a square.

$\overrightarrow{AB} = [-4, 2, 4]$ $\quad |\overrightarrow{AB}| = \sqrt{4^2 + 2^2 + 4^2} = 6$

$\overrightarrow{DC} = [-4, 2, 4]$

$\overrightarrow{AB} = \overrightarrow{DC}$ $\therefore$ AB and DC are equal in length and parallel

$\overrightarrow{AD} = [-4, -4, -2]$ $\quad |\overrightarrow{AD}| = \sqrt{4^2 + 4^2 + 2^2} = 6$ $\therefore$ AB and AD are equal in length

$\overrightarrow{BC} = [-4, -4, -2]$

$\overrightarrow{AD} = \overrightarrow{BC}$ $\therefore$ AD and BC are equal in length and parallel

$\overrightarrow{AB} \cdot \overrightarrow{AD} = 16 - 8 - 8 = 0$ $\therefore$ AB and AD are perpendicular

$\therefore$ ABCD is a square

(3 marks)

(c) Write an equation of the line through P $(-2, 9, 3)$ with direction normal to the plane.

$$\underset{\sim}{r} = [-2, 9, 3] + \lambda [1, -2, 2]$$

i.e. $\begin{cases} x = -2 + \lambda \\ y = 9 - 2\lambda \\ z = 3 + 2\lambda \end{cases}$

(2 marks)

(d) Find the point Q of intersection of the plane $x - 2y + 2z = -5$ and the line found in part (c).

Substituting the parametric equations of the line into the equation of the plane gives

$$(-2+\lambda) - 2(9-2\lambda) + 2(3+2\lambda) = -5$$

$$-14 + 9\lambda = -5$$

$$9\lambda = 9$$

$\lambda = 1$ $\therefore$ the point of intersection is $(-1, 7, 5)$

(3 marks)

(e) Determine whether Q is inside or outside the square.

Solving $\overrightarrow{OQ} = \overrightarrow{OA} + \lambda \overrightarrow{AB} + \mu \overrightarrow{AD}$ gives

$$[-1, 7, 5] = [3, 4, 0] + \lambda [-4, 2, 4] + \mu [-4, -4, -2]$$

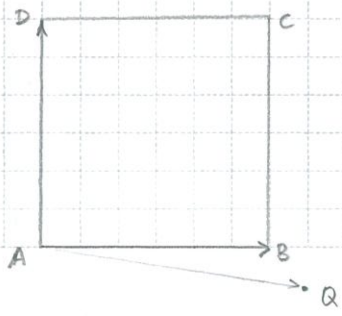
x: $-4\lambda - 4\mu = -4$

y: $2\lambda - 4\mu = 3$

$\Big\}$ $\lambda = 7/6$ and $\mu = -1/6$

z: $4\lambda - 2\mu = 5$ ✓ (this is consistent with the above values of λ and μ)

$\therefore$ Q is outside the square (to be inside the square $0 < \lambda < 1$ and $0 < \mu < 1$)



$\overrightarrow{AQ} = \frac{7}{6}\overrightarrow{AB} - \frac{1}{6}\overrightarrow{AD}$

(3 marks)

QUESTION 7 (5 marks)

Consider the vectors $\underset{\sim}{u} = 3\underset{\sim}{i} + 5\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

(a) Find the scalars *a* and *b* if $a(\underset{\sim}{u} + \underset{\sim}{v}) = 10\underset{\sim}{i} + (b-2)\underset{\sim}{j} + 2\underset{\sim}{k}$.

$a\left(5\underline{i} + 4\underline{j} + \underline{k}\right) = 10\underline{i} + (b-2)\underline{j} + 2\underline{k}$

$\underline{i}$: $5a = 10 \Rightarrow a = 2$

$\underline{j}$: $4a = b-2 \Rightarrow b = 10$

$\underline{k}$: $a = 2$ ✓ (consistent with the above values of a and b)

(3 marks)

(b) Find the projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$.

The projection vector of $\underline{u}$ onto $\underline{v}$ is $\left(\dfrac{\underline{u}\cdot\underline{v}}{|\underline{v}^2|}\right)\underline{v}$

$= \left(\dfrac{6-5-2}{2^2+1^2+2^2}\right)\left(2\underline{i} - \underline{j} + 2\underline{k}\right)$

$= \dfrac{-1}{9}\left(2\underline{i} - \underline{j} + 2\underline{k}\right)$

(2 marks)

END OF PART 2