Question 8 (15 marks)

(a) Consider the planes P_1 and P_2 that are defined by the equations below.

$$P_1: \ x+y+2z=4$$
$$P_2: \ 2x-y+z=8$$

(i) Show that P_1 and P_2 intersect at line l_1, which has the following parametric equations:

$$\begin{cases} x=4-t \\ y=-t \\ z=t \end{cases} \quad \text{where } t \text{ is a real parameter.}$$

Clearly state all row operations.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 2 & -1 & 1 & 8 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 3 & 0 \end{array}\right] 2R_1 - R_2$$

Let $z=t$ $\quad R_2 \Rightarrow 3y+3t=0$ $\quad R_1 \Rightarrow x - t + 2t = 4$

$3y=-3t$ $\quad x = 4 - t$

$y=-t$

(3 marks)

(ii) Show that the points $A(0,-4,4)$ and $B(1,-3,3)$ are on l_1.

$t=4 \Rightarrow \begin{cases} x=0 \\ y=-4 \\ z=4 \end{cases}$ $\quad t=3 \Rightarrow \begin{cases} x=1 \\ y=-3 \\ z=3 \end{cases}$

$\therefore A(0,-4,4)$ is on l_1 $\quad \therefore B(1,-3,3)$ is on l_1

(2 marks)

(iii) Show that the point $C(4,2,2)$ is on P_2.

$2(4)-(2)+(2)=8$

$\therefore C(4,2,2)$ is on P_2

(1 mark)

(b) Figure 6 shows P_1 and P_2, and the line l_1 where P_1 and P_2 intersect.

The normal to P_2 through C meets P_1 at the point D.

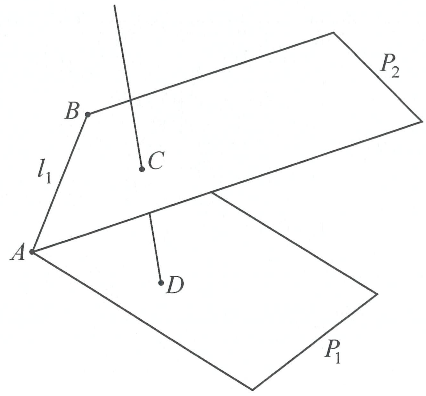

Figure 6

(i) Find the equation of the normal to P_2 through C.

$[2, -1, 1]$ is normal to P_2 $\therefore$ The equation of the normal to P_2 through C is

$\underline{r} = [4, 2, 2] + \lambda[2, -1, 1]$ ie. $\begin{cases} x = 4 + 2\lambda \\ y = 2 - \lambda \\ z = 2 + \lambda \end{cases}$

(2 marks)

(ii) Show that D has coordinates $(0, 4, 0)$.

Substituting the parametric equations above into the equation of P_1 gives

$(4 + 2\lambda) + (2 - \lambda) + 2(2 + \lambda) = 4$

$10 + 3\lambda = 4$

$\lambda = -2 \Rightarrow$ D has coordinates $(0, 4, 0)$

(2 marks)

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(iii) From part (a)(i), the parametric equations for l_1 are:

$$\begin{cases} x = 4 - t \\ y = -t \\ z = t \end{cases} \quad \text{where } t \text{ is a real parameter.}$$

Find the coordinates of the point on l_1 that is closest to $D(0, 4, 0)$.

Let P be the point on l_1 which is closest to D

$\therefore \vec{PD} \cdot [-1, -1, 1] = 0$

$[4-t, -t-4, t] \cdot [-1, -1, 1] = 0$

$t - 4 + t + 4 + t = 0$

$3t = 0$

$t = 0$

$\therefore$ P has coordinates $(4, 0, 0)$

(3 marks)

(iv) How much closer is D to P_2 than it is to l_1?

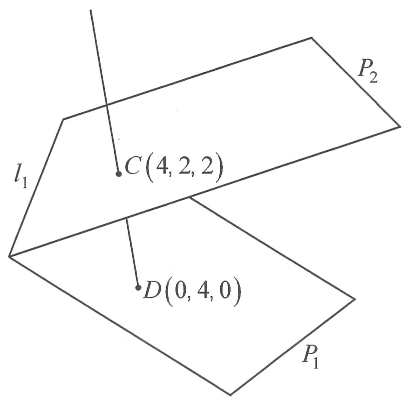

Figure 7

Distance from D to P_2 $= |\vec{CD}|$

$= \sqrt{4^2 + 2^2 + 2^2}$

$= \sqrt{24} = 2\sqrt{6}$ units $\simeq 4.90$ units

Distance from D to l_1 $= |\vec{DP}|$

$= \sqrt{4^2 + 4^2}$

$= \sqrt{32} = 4\sqrt{2}$ units $\simeq 5.66$ units

Closer by $4\sqrt{2} - 2\sqrt{6}$ units $\simeq 0.758$ units

(2 marks)