Question 8 (15 marks)

Points $A(5,-1,-3)$, $B(5,-3,-1)$, and $D(1,-1,1)$ are on the circumference of a circle with centre $C(3,-1,-1)$ on the plane P_1, as shown in Figure 8.

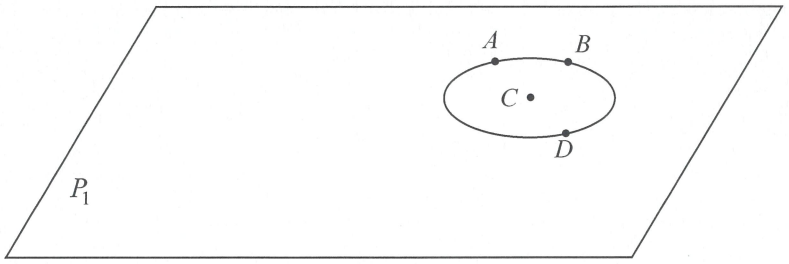

Figure 8

(a) (i) Find $\overrightarrow{BA} \times \overrightarrow{BD}$.

$\overrightarrow{BA} = [0, 2, -2]$

$\overrightarrow{BD} = [-4, 2, 2]$

$$\overrightarrow{BA} \times \overrightarrow{BD} = \begin{vmatrix} i & j & k \\ 0 & 2 & -2 \\ -4 & 2 & 2 \end{vmatrix} = [8, 8, 8]$$

(2 marks)

(ii) Hence show that the equation of plane P_1 is $x + y + z = 1$.

$[8, 8, 8]$ is normal to P_1 $\therefore [1, 1, 1]$ is normal to P_1

$\therefore$ The equation of P_1 is $x + y + z = (5) + (-1) + (-3)$

$x + y + z = 1$

(2 marks)

(b) (i) Show that AD is a diameter of the circle.

$\overrightarrow{AC} = [-2, 0, 2]$

$\overrightarrow{AD} = [-4, 0, 4]$

$\overrightarrow{AD} = 2\overrightarrow{AC}$

$\therefore$ AD is a diameter of the circle

(1 mark)

(ii) Find the radius of the circle.

$|\overrightarrow{AC}| = \sqrt{2^2 + 0^2 + 2^2}$

$= \sqrt{8}$ units

(1 mark)

(c) Point $E(8,-4,-3)$ is on the plane P_1.

Show that the parametric equations of the line through E and B are:

$$\begin{cases} x = 8 - 3t \\ y = -4 + t \\ z = -3 + 2t \end{cases}$$ where t is a real parameter.

$\overrightarrow{EB} = [-3, 1, 2]$

$\therefore$ The equation of the line through E and B is $\underline{r} = [8,-4,-3] + t[-3, 1, 2]$

i.e. $\begin{cases} x = 8 - 3t \\ y = -4 + t \\ z = -3 + 2t \end{cases}$

(2 marks)

(d) The equation of the circle on P_1 with centre C and passing through A, B, and D is:

$$(x-3)^2 + (y+1)^2 + (z+1)^2 = 8.$$

Show that the line through E and B intersects the circle again at $X\left(\frac{11}{7}, -\frac{13}{7}, \frac{9}{7}\right)$.

Substituting the parametric equations of the line into the equation of the circle gives

$(5-3t)^2 + (t-3)^2 + (2t-2)^2 = 8$

$9t^2 - 30t + 25 + t^2 - 6t + 9 + 4t^2 - 8t + 4 = 8$

$14t^2 - 44t + 30 = 0$

$7t^2 - 22t + 30 = 0$

$(7t-15)(t-1) = 0$

$t = \frac{15}{7}$ or $t = 1$

$t = 1 \Rightarrow$ line intersects the circle at $B(5, -3, -1)$

$t = \frac{15}{7} \Rightarrow$ line intersects the circle at $X\left(\frac{11}{7}, \frac{-13}{7}, \frac{9}{7}\right)$

(4 marks)

(e) Find the arc length BX.

$\overrightarrow{CB} = [2, -2, 0]$

$\overrightarrow{CX} = \left[\frac{-10}{7}, \frac{-6}{7}, \frac{16}{7}\right]$

$\angle BCX = \cos^{-1}\left(\frac{-8}{\sqrt{8}.\sqrt{392}}\right) = 98.2^\circ$

$\therefore$ Arc length $BX = \frac{98.2}{360} \times 2\pi \times \sqrt{8} = 4.85$ units

(3 marks)