

**Question 4** (9 marks)

Consider three planes in space that are defined by the following system of equations:

$$P_1: x - 2y + z = 4$$

$$P_2: x + 3y - z = 0$$

$$P_3: 2x - y + az = a$$

where  $a$  is a real constant.

(a) Write the system of equations in augmented matrix form.

$$\left[ \begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 1 & 3 & -1 & 0 \\ 2 & -1 & a & a \end{array} \right]$$

(1 mark)

(b) Clearly stating all row operations, show that the augmented matrix reduces to

$$\left[ \begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 5 & -2 & -4 \\ 0 & 0 & (4-5a) & (28-5a) \end{array} \right].$$

$$\sim \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 5 & -2 & -4 \\ 0 & 3 & a-2 & a-8 \end{array} \right] \begin{array}{l} R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$
  

$$\sim \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 5 & -2 & -4 \\ 0 & 0 & 4-5a & 28-5a \end{array} \right] 3R_2 - 5R_3$$

(3 marks)

(c) (i) Solve the system of equations for  $a = 0$ .

$$a = 0 \Rightarrow \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 5 & -2 & -4 \\ 0 & 0 & 4 & 28 \end{array} \right] \quad \begin{array}{l} R_3 \Rightarrow z = 7 \\ R_2 \Rightarrow 5y - 14 = -4 \\ \quad 5y = 10 \\ \quad y = 2 \\ R_1 \Rightarrow x - 4 + 7 = 4 \\ \quad x = 1 \end{array}$$

(2 marks)

(ii) Solve the system of equations for  $a = \frac{4}{5}$ .

$$a = \frac{4}{5} \Rightarrow \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 5 & -2 & -4 \\ 0 & 0 & 0 & 24 \end{array} \right]$$

$R_3 \Rightarrow$  this system of equations has no solution

(1 mark)

(iii) Which *one* of the following figures best represents the configuration of the three planes when  $a = \frac{4}{5}$ ? Justify your answer.

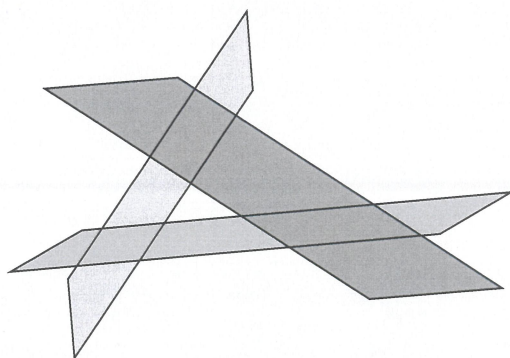


Figure 2

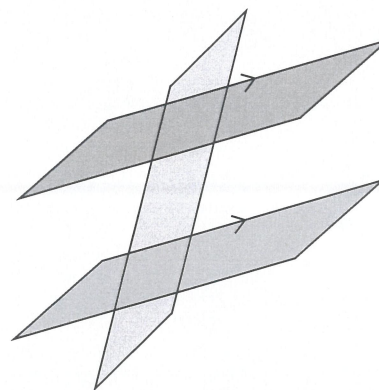


Figure 3

Comparing the left hand sides of the equations of  $P_1$ ,  $P_2$ , and  $P_3$ , we see that none are parallel  
 $\therefore$  Figure 2 best represents the configuration of the three planes when  $a = \frac{4}{5}$

(2 marks)