PART 2 (Questions 11 to 15)

(75 marks)

Question 11 (15 marks)

(a) Calculate the vector (cross) product $[1,-1,-1]\times[1,0,1]$.

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = [-1, -2, 1]$$

(1 mark)

(b) Consider the planes P_1 and P_2 that are defined by the following equations:

$$P_1 : x - y - z = 4$$
$$P_2 : \quad x + z = 9.$$

Figure 8 shows P_1, P_2, and the line l_1, where P_1 and P_2 intersect.

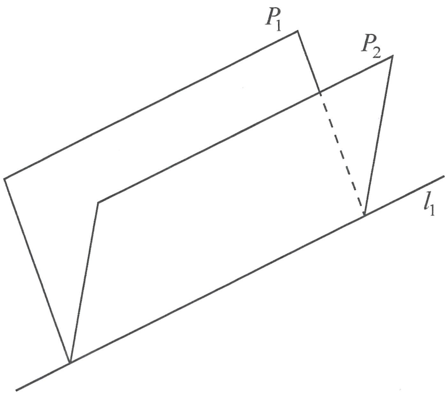

Figure 8

(i) Show that the point $X(9, 5, 0)$ is on both planes.

P_1 : $(9) - (5) - (0) = 4$ $\therefore X$ is on P_1

P_2 : $(9) + (0) = 9$ $\therefore X$ is on P_2

(1 mark)

(ii) Hence or otherwise, show that l_1 has the following parametric equations:

$$\begin{cases} x = 9 - t \\ y = 5 - 2t \\ z = t \end{cases} \quad \text{where } t \text{ is real.}$$

$[1, -1, -1]$ is normal to plane P_1

$[1, 0, 1]$ is normal to plane P_2

$\therefore [1, -1, -1] \times [1, 0, 1] = [-1, -2, 1]$ is parallel to line l_1

$\therefore$ The equation of l_1 is $\underline{r} = [9, 5, 0] + t[-1, -2, 1]$

i.e. $\begin{cases} x = 9 - t \\ y = 5 - 2t \\ z = t \end{cases}$

(2 marks)

(c) Consider the line l_2, which has the following parametric equations:

$$\begin{cases} x = 3 + 3s \\ y = -s \\ z = 3 \end{cases} \quad \text{where } s \text{ is real.}$$

(i) (1) Show that l_2 intersects l_1.

If l_2 intersects l_1 then $\begin{cases} 9 - t = 3 + 3s & ① \\ 5 - 2t = -s & ② \\ t = 3 & ③ \end{cases}$

③ $\Rightarrow$ $t = 3$

② $\Rightarrow$ $-1 = -s$

$s = 1$

① $\Rightarrow$ $6 = 6$ ✓ (consistent) $\therefore l_2$ intersects l_1

(2 marks)

(2) Find Y, the point where l_1 and l_2 intersect.

$Y = (6, -1, 3)$

(1 mark)

The line l_2 lies on the plane P_3.

Plane P_3 intersects P_1 and P_2 along the common line l_1, as shown in Figure 9.

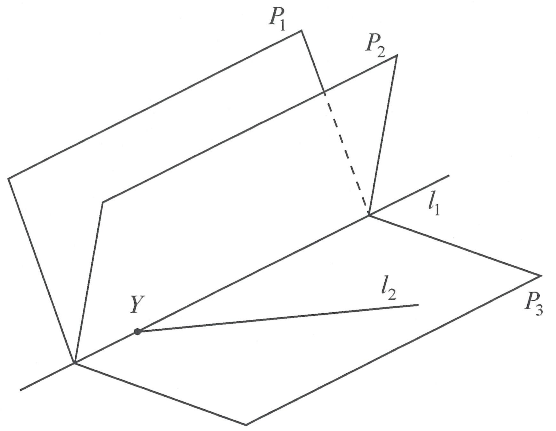

Figure 9

(ii) Show that the equation of P_3 is $x+3y+7z=24$.

$[-1, -2, 1] \times [3, -1, 0] = [1, 3, 7]$ is normal to P_3 and $Y(6, -1, 3)$ is on P_3

$\therefore$ The equation of P_3 is given by $x + 3y + 7z = (6) + 3(-1) + 7(3)$

$x + 3y + 7z = 24$

(3 marks)

(d) The line l_3 is parallel to l_2, as shown in Figure 10.

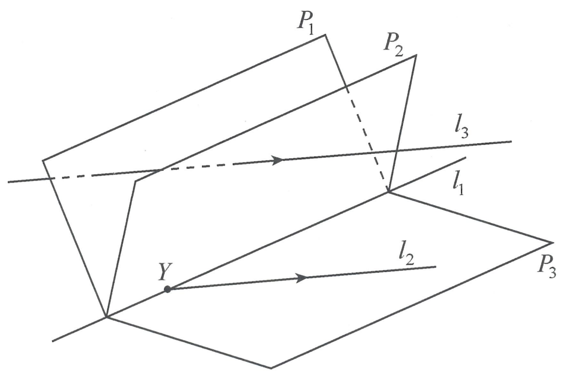

Figure 10

(i) Line l_3 passes through the origin.

Write an equation for l_3.

$\underline{r} = [0, 0, 0] + \lambda[3, -1, 0]$ ie. $\begin{cases} x = 3\lambda \\ y = -\lambda \\ z = 0 \end{cases}$

(1 mark)

(ii) Verify that l_3 does ***not*** lie on P_3.

Substituting the parametric equations of l_3 into P_3 gives

$(3\lambda) + 3(-\lambda) + 7(0) = 0$

$\neq 24$ $\therefore l_3$ does not lie on P_3

(1 mark)

(e) Particles are fired from a source located at the origin and travel along l_3.

(i) If the particles travel at a constant speed of $\sqrt{10}$ units/second, show that the particles pass through P_1, 1 second after they have been fired.

$|[3, -1, 0]| = \sqrt{9 + 1 + 0} = \sqrt{10}$

$\therefore \underline{r} = [0, 0, 0] + \frac{\cancel{\sqrt{10}}t}{\cancel{\sqrt{10}}}[3, -1, 0]$

Substituting into P_1 gives $(3t) - (-t) - (0) = 4$

$4t = 4$

$t = 1$

(2 marks)

(ii) How many more seconds elapse before the particles pass through P_2?

Substituting into P_2 gives $(3t) + (0) = 9$

$3t = 9$

$t = 3$

$\therefore$ 2 more seconds elapse before the particles pass through P_2

(1 mark)