Stage 2 Mathematical Methods

Differential Calculus Test (Exponential and Logarithmic Functions)

Topic 1: Subtopics 1.3, 4.1, 4.2, 4.3

Total Marks – 38

(Calculator and one A4 page of hand written notes permitted)

QUESTION 1 (9 marks)

Differentiate the following. There is no need to simplify your answers.

(a) $P = \frac{101}{1+5e^{-0.004t}}$

$P = 101.(1+5e^{-0.004t})^{-1}$

$\frac{dP}{dt} = -101(1+5e^{-0.004t})^{-2}.5e^{-0.004t}.-0.004$

(3 marks)

(b) $f(x) = x^3e^{2x^2-4}$

$f'(x) = 3x^2.e^{2x^2-4} + x^3.e^{2x^2-4}.4x$

(3 marks)

(c) $y = \ln(x\sqrt{1-2x})$

$y = \ln\left(x.(1-2x)^{\frac{1}{2}}\right)$

$= \ln x + \ln(1-2x)^{\frac{1}{2}}$

$= \ln x + \frac{1}{2}\ln(1-2x)$

$\frac{dy}{dx} = \frac{1}{x} + \frac{-2}{2(1-2x)}$

(3 marks)

QUESTION 2 (12 marks)

Let $f(x) = 1 - 2\ln x$. Consider the graph of $y = f(x)$.

(a) Find the *exact* values of the axis intercepts of this graph (if they exist).

$f(0)$ is undefined

$\therefore$ There is no y-intercept

$f(x) = 0 \Rightarrow 1 - 2\ln x = 0$

$-2\ln x = -1$

$\ln x = \frac{1}{2}$

$x = e^{1/2}$

$\therefore$ There is one x-intercept with coordinates $(\sqrt{e}, 0)$

(3 marks)

(b) Show that the equation of the tangent to the graph at $x = e$ is given by $2x + ey = e$.

x-coordinate of point of contact $= e$ (given)

y-coordinate of point of contact $= f(e)$

$= 1 - 2\ln e$

$= -1$

$f'(x) = -\frac{2}{x}$

$\therefore$ Gradient of the tangent $= -\frac{2}{e}$

$\therefore$ Equation of tangent is $y + 1 = -\frac{2}{e}(x - e)$

$ey + e = -2x + 2e$

$2x + ey = e$

(3 marks)

(c) Find the coordinates of the point at which this tangent cuts the y-axis.

Substituting $x=0$ into the equation of the tangent gives $ey = e$

$\therefore y = 1$

$\therefore$ The coordinates are $(0, 1)$

(1 mark)

(d) For what values of x is $f(x)$ defined?

$x > 0$

(1 mark)

(e) Show that the graph of $y = f(x)$ is concave upwards for all values of x for which it is defined.

$f(x)$ is concave up whenever $f''(x) > 0$

$f'(x) = -2x^{-1}$

$\therefore f''(x) = 2x^{-2}$

$= \frac{2}{x^2}$

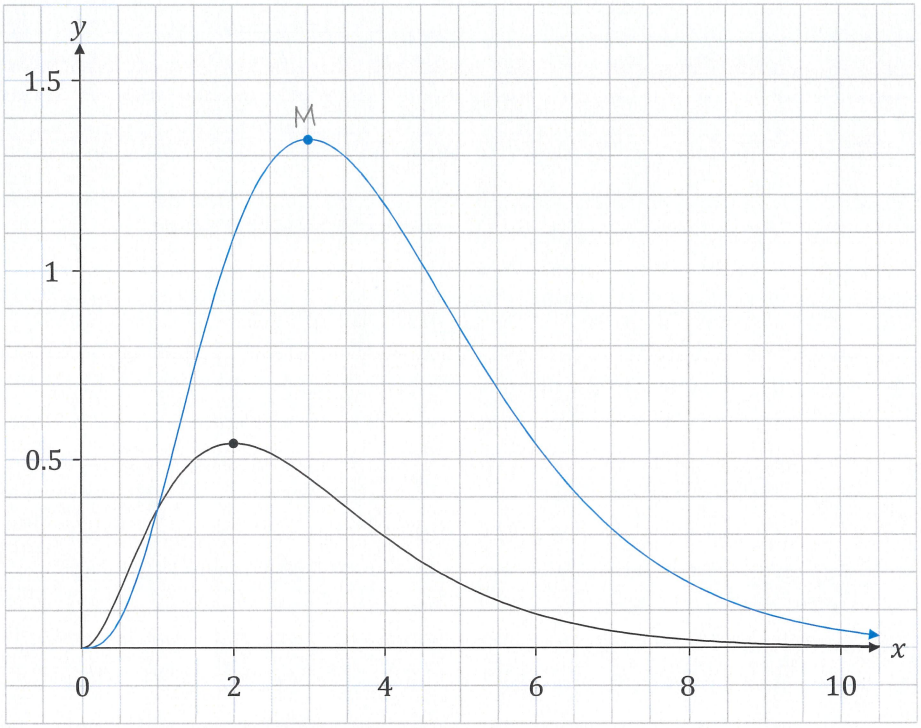
> 0 for all $x > 0$

$\therefore y = f(x)$ is concave upwards for all values of x for which it is defined

(2 marks)

QUESTION 3 (9 marks)

The graph of $y = x^2e^{-x}$ is shown below, for $x \geq 0$:



(a) Use your graphics calculator to find the coordinates of the maximum visible in the graph above.

(2, 0.541)

(1 mark)

(b) On the axes above, carefully sketch the graph of $y = x^3e^{-x}$.

Accurately plot, and label with an M, the maximum of this graph.

(2 marks)

(c) Use your graphics calculator to complete the following table:

function	$y = x^2e^{-x}$	$y = x^3e^{-x}$	$y = x^4e^{-x}$	$y = x^5e^{-x}$
x-coordinate of maximum	2	3	4	5

(2 marks)

(d) Make a conjecture about the x-coordinate of the maximum of $y = x^ne^{-x}$ where n is a positive integer.

The x-coordinate of the maximum of $y = x^n.e^{-x}$ is equal to n

(1 mark)

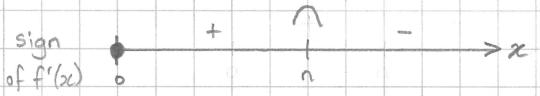
(e) Prove or disprove the conjecture you made in part (d).

$f(x) = x^n . e^{-x}$

$f'(x) = nx^{n-1} . e^{-x} - x^n . e^{-x}$

$= x^{n-1} . e^{-x}(n - x)$

$f'(x) = 0 \Rightarrow x = 0$ or $\underline{x = n}$



(3 marks)

QUESTION 4 (8 marks)

A tank initially contains 80 L of liquid. The volume, V, of liquid contained inside the tank after t minutes is given by $V = V_0 e^{kt}$ where V_0 is the original volume of the liquid

(a) Show that $\frac{dV}{dt} = kV$.

$V = 80e^{kt}$

$\therefore \frac{dV}{dt} = k.80e^{kt}$

$= k.V$

(1 mark)

(b) If the volume is **INCREASING** at a rate of 5 L per minute at the instant the volume reaches 250 L, calculate the value of k.

$\frac{dV}{dt} = +5$ when $V = 250$

$\therefore 5 = k.250$

$k = 0.02$

(2 marks)

(c) What volume of liquid is contained within the tank after 10 minutes?

$V(t) = 80e^{0.02t}$

$\therefore V(10) = 97.7L$

(1 mark)

(d) Calculate the rate at which the volume is changing after 10 minutes?

$V'(t) = 0.02V$

$\therefore V'(10) = 1.95$ ie. the volume is increasing at 1.95L/min

(1 mark)

(e) *Exactly* how long will it take for the volume to reach 200 L?

$V(t) = 200 \Rightarrow 80e^{0.02t} = 200$

$e^{0.02t} = 2.5$

$0.02t = \ln 2.5$

$t = 50 \ln 2.5$ mins

(3 marks)