Question 15 (15 marks)

(a) Using integration by parts, show that $\int \ln x \, dx = x \ln x - x + c$, where c is a constant, for $x > 0$.

(2 marks)

(b) Consider the function $y = e^{x^2} - 15$, where $y \geq 0$, as shown in Figure 16.

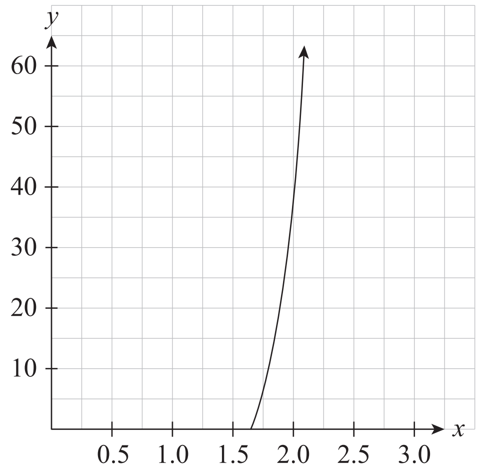

Figure 16

(i) This curve, y, is rotated about the y-axis bounded by the lines $y = 0$ and $y = h$ (where $h > 0$), forming a solid with volume V.

Show that $V = \pi\left[(h+15)(\ln(h+15)-1) - 15(\ln 15 - 1)\right]$.

(3 marks)

The volume found in part (b)(i) represents the volume of a container that holds water.

The container, initially full of water, begins leaking from a small hole in the base of the container. The depth of water in the container, h cm, varies with time t, measured in seconds.

(ii) Show that the rate at which water leaks from the container is given by

$$\frac{dV}{dt} = \frac{dh}{dt}\pi \ln(h+15).$$

(3 marks)

(c) Another formula for the rate at which the container leaks is $\frac{dV}{dt} = -0.98\sqrt{h}$.

(i) Using part (b)(ii), show that $\int 0.98 dt = -\pi \int \frac{\ln(h+15)}{\sqrt{h}} dh$.

(2 marks)

(ii) Suppose the initial depth of the water in the container is 35 cm.

Using the result from part (c)(i), explain clearly why the time taken for the container to empty, t_e, is

$$t_e = \frac{\pi}{0.98}\int_0^{35} \frac{\ln(h+15)}{\sqrt{h}}\,dh.$$

(2 marks)

(iii) Hence find the time taken for the container to empty, correct to the nearest second.

(1 mark)

(d) Now consider the situation where the container is initially empty. Water is pumped into the container at a constant rate of 2.43 cm^3 per second.

At the same time, water leaks from the container at a rate given by $\frac{dV}{dt} = -0.98\sqrt{h}$.

Determine the depth at which the level of water in the container is not increasing and not decreasing.

(2 marks)