Question 7 (10 marks)

(a) Using the fact that $\sin^3 x = \sin^2 x \sin x$ show that

$$\int \sin^3 x \, dx = -\cos x + \frac{1}{3}\cos^3 x + c, \text{ where } c \text{ is a constant.}$$

(3 marks)

(b) Figure 5 shows the graph of $f(x) = \sin^3 x$ for $0 \leq x \leq \frac{\pi}{2}$ and the graph of $y = x$ for $x \geq 0$.

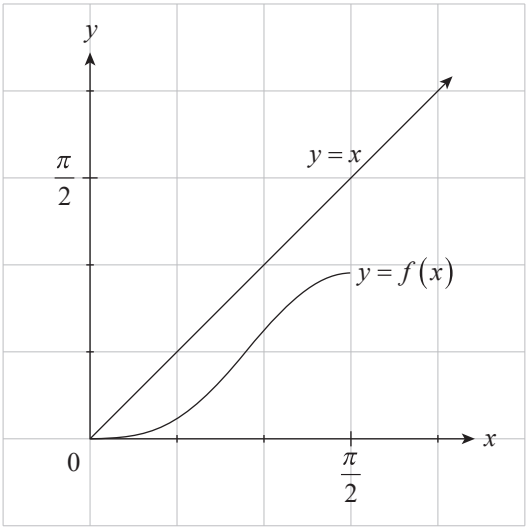

Figure 5

(i) Find $f\left(\frac{\pi}{2}\right)$.

(1 mark)

(ii) Explain why the function $f(x)$ has an inverse function.

(1 mark)

(iii) On Figure 5, draw the graph of the inverse function, $f^{-1}(x)$, using symmetry about the line $y = x$. (1 mark)

(c) Use parts (a) and (b) to show that $\int_0^1 f^{-1}(x)\,\mathrm{d}x = \frac{\pi}{2} - \frac{2}{3}$.

(4 marks)