Question 6 (6 marks)

(a) Use integration by parts to show that

$$\int \arctan x \, dx = x \arctan x - \frac{1}{2}\ln\left(x^2+1\right) + c, \text{ where } c \text{ is a constant.}$$

Let $u = \arctan x$ and $v' = 1$

Then $u' = \dfrac{1}{1+x^2}$ and $v = x$

$\therefore \int \arctan x \, dx = x.\arctan x - \frac{1}{2}\int \frac{2x}{1+x^2}\, dx = x.\arctan x - \frac{1}{2}\ln(x^2+1) + c$

(2 marks)

(b) Consider the graph of $f(x) = \sqrt{\arctan x}$ for $x \geq 0$, shown in Figure 4.

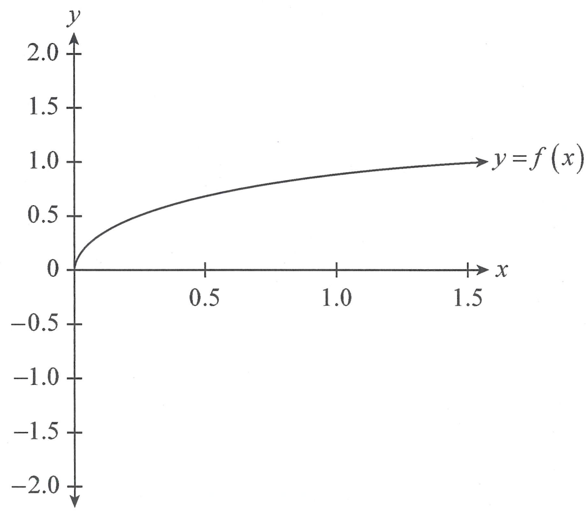

Figure 4

Consider rotating the graph of $f(x)$ about the x-axis between $x = 0$ and $x = 1$.

(i) Show that the volume of the solid that is obtained by this rotation is given by the equation below.

$$V = \pi \int_0^1 \arctan x \, dx$$

$V = \pi \int_0^1 f(x)^2 \, dx$

$= \pi \int_0^1 \arctan x \, dx$

(1 mark)

(ii) Hence find the ***exact*** volume of this solid.

$$V = \pi\left[x.\arctan x - \frac{1}{2}\ln(x^2+1)\right]_0^1 \quad \{\text{using part (a) above}\}$$

$$= \pi\left(1.\frac{\pi}{4} - \frac{1}{2}\ln 2\right) - \left(0 - \frac{1}{2}\cancel{\ln 1}^{\,0}\right)$$

$$= \frac{\pi^2}{4} - \frac{\pi}{2}\ln 2 \text{ units}^3$$

(3 marks)