Question 7 (10 marks)

(a) Using the fact that $\sin^3 x = \sin^2 x \sin x$ show that

$$\int \sin^3 x \, dx = -\cos x + \frac{1}{3}\cos^3 x + c \text{, where } c \text{ is a constant.}$$

$\int \sin^3 x \, dx = \int (1 - \cos^2 x) . \sin x \, dx$

$= \int \sin x \, dx - \int \cos^2 x . \sin x \, dx$ $\quad \left\{ \text{Let } u = \cos x \text{ then } \frac{du}{dx} = -\sin x \right\}$

$= -\cos x + \int u^2 . \frac{du}{dx} . dx$

$= -\cos x + \frac{u^3}{3} + c$

$= -\cos x + \frac{1}{3}\cos^3 x + c$

(3 marks)

(b) Figure 5 shows the graph of $f(x) = \sin^3 x$ for $0 \le x \le \frac{\pi}{2}$ and the graph of $y = x$ for $x \ge 0$.

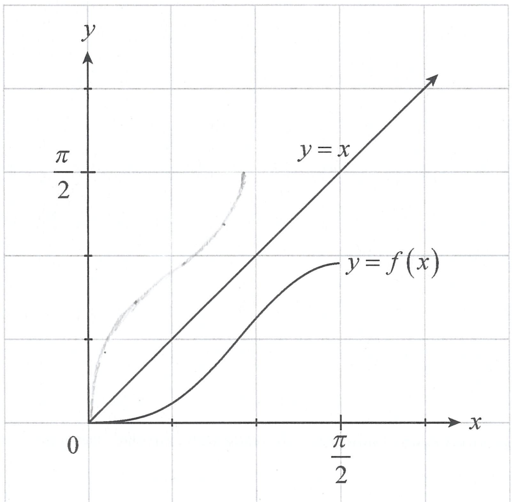

Figure 5

(i) Find $f\left(\frac{\pi}{2}\right)$.

$f\left(\frac{\pi}{2}\right) = 1$

(1 mark)

(ii) Explain why the function $f(x)$ has an inverse function.

f(x) is a one-to-one function

$\therefore$ f(x) has an inverse function

(1 mark)

(iii) On Figure 5, draw the graph of the inverse function, $f^{-1}(x)$, using symmetry about the line $y = x$.

(1 mark)

(c) Use parts (a) and (b) to show that $\int_0^1 f^{-1}(x)\,dx = \frac{\pi}{2} - \frac{2}{3}$.

$$\int_0^1 f^{-1}(x)\,dx = \int_0^{\pi/2} 1 - f(x)\,dx \qquad \{\text{using symmetry}\}$$

$$= \int_0^{\pi/2} 1 - \sin^3 x\,dx$$

$$= \left[x + \cos x - \frac{1}{3}\cos^3 x \right]_0^{\pi/2}$$

$$= \left(\frac{\pi}{2} + 0 - 0 \right) - \left(0 + 1 - \frac{1}{3} \right)$$

$$= \frac{\pi}{2} - \frac{2}{3}$$

(4 marks)