Question 7 (7 marks)

(a) Verify that $\dfrac{x^2}{x+1} = (x-1) + \dfrac{1}{x+1}$.

$$\frac{x^2}{x+1} = \frac{x^2 - 1 + 1}{x+1}$$

$$= \frac{\cancel{(x+1)}(x-1)}{\cancel{x+1}} + \frac{1}{x+1}$$

$$= (x-1) + \frac{1}{x+1}$$

(1 mark)

(b) Use integration by parts to show that, for $x > -1$:

$$\int x \ln(x+1)\,dx = \frac{x^2}{2}\ln(x+1) - \frac{1}{4}(x-1)^2 - \frac{1}{2}\ln(x+1) + c$$

where c is a constant.

Let $u = \ln(x+1)$ and $v' = x$

Then $u' = \dfrac{1}{x+1}$ and $v = \dfrac{x^2}{2}$

$$\therefore \int x . \ln(x+1)\,dx = \frac{x^2}{2} . \ln(x+1) - \int \frac{x^2}{2(x+1)}\,dx$$

$$= \frac{x^2}{2} . \ln(x+1) - \frac{1}{2}\int x - 1 + \frac{1}{x+1}\,dx \qquad \{\text{using (a) above}\}$$

$$= \frac{x^2}{2} . \ln(x+1) - \frac{1}{2}\left(\frac{x^2}{2} - x + \ln(x+1) + k\right)$$

$$= \frac{x^2}{2} . \ln(x+1) - \frac{1}{4}(x^2 - 2x + 1) - \frac{1}{2} . \ln(x+1) + c$$

$$= \frac{x^2}{2} . \ln(x+1) - \frac{1}{4}(x-1)^2 - \frac{1}{2}\ln(x+1) + c$$

(3 marks)

(c) (i) Figure 8 shows the graph of $f(x) = x\ln(x+1)$ for $x > -1$.

On the same axes, sketch the graph of $g(x) = x\left|\ln(x+1)\right|$ for $x > -1$.

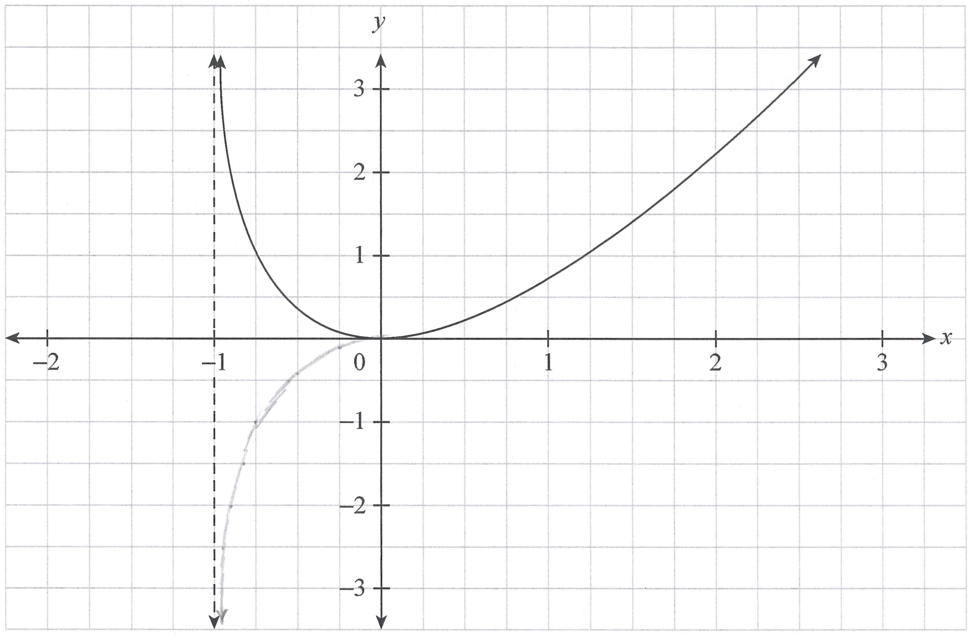

Figure 8

(1 mark)

(ii) Using the information in part (b), find the exact value of $\int_{-\frac{1}{2}}^{0} g(x)\,\mathrm{d}x$.

$$\int_{-1/2}^{0} g(x)\,dx = \int_{-1/2}^{0} -x.\ln(x+1)\,dx$$

$$= \left[-\frac{x^2}{2}.\ln(x+1) + \frac{1}{4}(x-1)^2 + \frac{1}{2}.\ln(x+1)\right]_{-1/2}^{0}$$

$$= \left(0 + \frac{1}{4} + 0\right) - \left(-\frac{1}{8}\ln\frac{1}{2} + \frac{9}{16} + \frac{1}{2}\ln\frac{1}{2}\right)$$

$$= -\frac{5}{16} - \frac{3}{8}\ln\frac{1}{2}$$

$$= \frac{-5}{16} + \frac{3}{8}\ln 2$$

(2 marks)