Question 13 (15 marks)

Figure 12 shows the graph of $g(x)=\arcsin(x^2)$ for $-1\le x\le 1$.

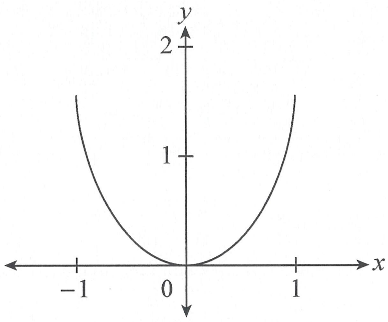

Figure 12

(a) (i) Explain why $g(x)$ does not have an inverse.

(1 mark)

(ii) Explain why $f(x)=\arcsin(x^2)$ for $0\le x\le 1$ does have an inverse.

(1 mark)

(b) Figure 13 shows the graph of $y=f(x)$.

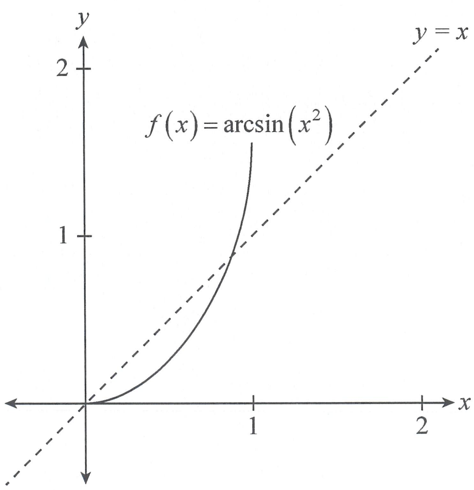

Figure 13

(i) On the axes in Figure 13, sketch the graph of $f^{-1}(x)$. (2 marks)

(ii) Write the exact domain of $f^{-1}(x)$.

(1 mark)

(iii) Find the equation of $f^{-1}(x)$.

(2 marks)

(c) If $y = \arcsin\left(x^2\right)$, then $x^2 = \sin y$.

Hence use implicit differentiation to show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x}{\sqrt{1-x^4}}$.

(3 marks)

(d) (i) Use integration by parts to show that

$$\int x \arcsin\left(x^2\right) dx = \frac{1}{2}x^2 \arcsin\left(x^2\right) + \frac{1}{2}\sqrt{1-x^4} + c$$

where c is a constant.

Let $u = \arcsin(x^2)$ and $v' = x$

Then $u' = \dfrac{2x}{\sqrt{1-x^4}}$ and $v = \dfrac{x^2}{2}$ {using part (c) above}

$\therefore \int x.\arcsin(x^2)\,dx = \dfrac{x^2}{2}.\arcsin(x^2) - \int \dfrac{x^3}{\sqrt{1-x^4}}\,dx$

$= \dfrac{x^2}{2}.\arcsin(x^2) + \dfrac{1}{4}\int \dfrac{-4x^3}{\sqrt{1-x^4}}\,dx$ Let $U = 1-x^4$ then $\dfrac{dU}{dx} = -4x^3$

$= \dfrac{x^2}{2}.\arcsin(x^2) + \dfrac{1}{4}\int U^{-1/2}.\dfrac{dU}{dx}.dx$

$= \dfrac{x^2}{2}.\arcsin(x^2) + \dfrac{1}{4}.\dfrac{2}{1}U^{1/2} + c$

$= \dfrac{x^2}{2}.\arcsin(x^2) + \dfrac{1}{2}\sqrt{1-x^4} + c$

(3 marks)

(ii) Hence find the exact value of $\int_0^1 x \arcsin\left(x^2\right) dx$.

$\therefore \int_0^1 x.\arcsin(x^2)\,dx = \left[\dfrac{x^2}{2}.\arcsin(x^2) + \dfrac{1}{2}\sqrt{1-x^4}\right]_0^1$

$= \left(\dfrac{1}{2}.\dfrac{\pi}{2} + \dfrac{1}{2}.\sqrt{0}\right) - \left(0 + \dfrac{1}{2}\sqrt{1}\right)$

$= \dfrac{\pi}{4} - \dfrac{1}{2}$

(2 marks)