Solutions

Stage 2 Specialist Mathematics

Integration Techniques and Applications Test

Topic 5: Subtopics 5.1, 5.2

Total Marks - 28

(Calculator and one A4 page of handwritten notes permitted.)

QUESTION 1 (8 marks)

Select the relevant integration rule or technique to integrate the following:

(a) $\int \sin^2(x)\,dx$

$= \int \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$

$= \frac{x}{2} - \frac{1}{4}\sin 2x + c$

(2 marks)

(b) $\int \frac{-5}{\sqrt{9-x^2}}\,dx$

$= 5\int \frac{-1}{\sqrt{3^2-x^2}}\,dx$

$= 5\arccos\left(\frac{x}{3}\right) + c$

(2 marks)

(c) $\int \frac{12}{1+16x^2}\,dx$

$= \int \frac{48 \times \frac{1}{4}}{16\left(\frac{1}{4^2}+x^2\right)}\,dx$

$= 3\arctan(4x) + c$

(2 marks)

(d) $\int \frac{x}{\sqrt{4-x^2}}\,dx$

$= -\frac{1}{2}\int u^{-\frac{1}{2}} \cdot \frac{du}{\cancel{dx}} \cdot \cancel{dx}$ where $u = 4-x^2$ and $\frac{du}{dx} = -2x$

$= -u^{\frac{1}{2}} + c$

$= -\sqrt{4-x^2} + c$

(2 marks)

QUESTION 2 (6 marks)

(a) Find the values of A and B such that $\frac{A}{x-3} - \frac{B}{(x+3)(x-3)} = \frac{4x}{x^2-9}$

$A(x+3) - B = 4x$ {multiplying the equation by $(x+3)(x-3)$}

$\therefore A = 4$ {comparing coefficients of x

$B = 12$ {comparing constant terms

(2 marks)

(b) Hence or otherwise, find $\int \frac{12}{(x+3)(x-3)} dx$

$\therefore \int \frac{4}{x-3} - \frac{12}{(x+3)(x-3)} dx = \int \frac{4x}{x^2-9} dx$

$\therefore \int \frac{12}{(x+3)(x-3)} dx = \int \frac{4}{x-3} - \frac{4x}{x^2-9} dx$

$= 4\ln|x-3| - 2\int \frac{1}{u} \frac{du}{\cancel{dx}} \cancel{dx}$ where $u = x^2-9$ and $\frac{du}{dx} = 2x$

$= 4\ln|x-3| - 2\ln|u| + c$

$= 4\ln|x-3| - 2\ln|x^2-9| + c$

(4 marks)

QUESTION 3 (8 marks)

(a) (i) Use integration by parts to find $\int xe^{2x}\,dx$.

Let $u = x$ and $v' = e^{2x}$

Then $u' = 1$ and $v = \frac{1}{2}e^{2x}$

$$\therefore \int xe^{2x}\,dx = \frac{x}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx$$

$$= \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + c$$

(3 marks)

(ii) Use integration by parts to show that $\int x^2e^{2x}\,dx = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c$, where c is a constant.

Let $u = x^2$ and $v' = e^{2x}$

Then $u' = 2x$ and $v = \frac{1}{2}e^{2x}$

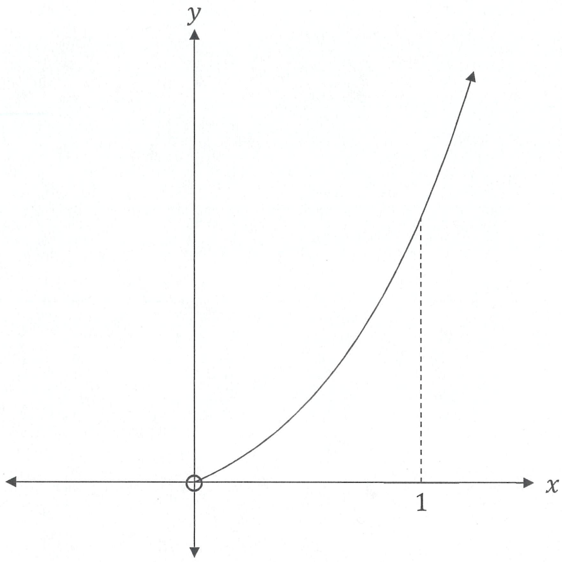
$$\therefore \int x^2e^{2x}\,dx = \frac{x^2}{2}e^{2x} - \int xe^{2x}\,dx$$

$$= \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + c$$ {using the result from (a)(i)}

(2 marks)

(b) Let $f(x) = xe^x$.

The graph of $y = f(x)$ for $x \geq 0$ is shown below.



Find the exact volume of the solid obtained when the region bounded by the graph of $f(x)$ on the interval $[0, 1]$ is rotated about the x-axis.

$V = \pi \int_0^1 x^2 e^{2x}\, dx$

$= \pi \left[\frac{x^2}{2} e^{2x} - \frac{x}{2} e^{2x} + \frac{1}{4} e^{2x} \right]_0^1$

$= \pi \left(\cancel{\frac{1}{2} e^2} - \cancel{\frac{1}{2} e^2} + \frac{1}{4} e^2 \right) - \pi \left(\frac{1}{4} \right)$

$= \frac{\pi}{4}(e^2 - 1)$ units^3

(3 marks)

QUESTION 4 (6 marks)

Let $f(x) = \frac{2x}{x^2+5}$.

(a) Find $\int \frac{2x}{x^2+5} dx$.

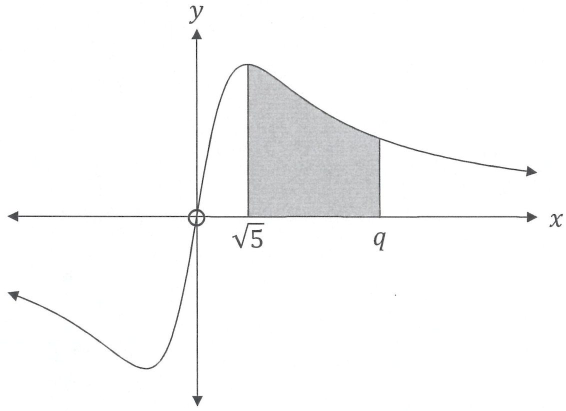
$$\int \frac{2x}{x^2+5} dx = \int \frac{1}{u} \frac{du}{dx} . dx \quad \text{where } u = x^2+5 \text{ and } \frac{du}{dx} = 2x$$

$$= \ln|u| + c$$

$$= \ln|x^2+5| + c$$

(2 marks)

The following diagram shows part of the graph of $y = f(x)$.



(b) The shaded region is enclosed by the graph of f, the x-axis, and the lines $x = \sqrt{5}$ and $x = q$. This region has an area of $\ln 7$. Find the value of q.

$$\int_{\sqrt{5}}^{q} \frac{2x}{x^2+5} dx = \ln 7$$

$$\therefore \left[\ln(x^2+5) \right]_{\sqrt{5}}^{q} = \ln 7$$

$$\ln(q^2+5) - \ln 10 = \ln 7$$

$$\ln(q^2+5) = \ln 7 + \ln 10$$

$$\ln(q^2+5) = \ln 70$$

$$q^2+5 = 70$$

$$q^2 = 65$$

$$q = \sqrt{65} \quad \text{since } q > \sqrt{5}$$

(4 marks)