Question 10 (9 marks)

For positive integer values of n, the function $f(x) = kx^n(1-x)$ forms a probability density function on the interval $0 \le x \le 1$ for a certain integer value of k. For this value of k, $f(x) \ge 0$ for $0 \le x \le 1$.

(a) For $n = 1$, algebraically find the value of k such that $f(x)$ forms a probability density function for $0 \le x \le 1$.

(3 marks)

(b) (i) Find the area under the curve of $y = x^2(1-x)$, for $0 \le x \le 1$.

(1 mark)

(ii) Hence find the value of k such that $f(x) = kx^2(1-x)$ forms a probability density function for $0 \le x \le 1$.

(1 mark)

After considering several more values of n, the following conjecture is made:

'In order for $f\left(x\right)=kx^{n}\left(1-x\right)$ to form a probability density function for $0\leq x\leq 1$, $k=\left(n+1\right)\left(n+2\right)$'.

(c) Prove or disprove this conjecture.

(4 marks)