Question 10 (8 marks)

The service life of an item is defined as an item's total life in use from the point of sale to the point of discard. A probability density function often used to model the service life of items is of the form

$$p_k(x) = \frac{k}{2}\left(\frac{x}{2}\right)^{k-1} e^{-\left(\frac{x}{2}\right)^k}$$

where x is measured in years for $x \geq 0$, and k is a constant such that $k \geq 1$. The value of k is determined by the failure rate of the item over time.

(a) Consider the case where $k = 2$, resulting in the probability density function

$$p_2(x) = \frac{x}{2} e^{-\left(\frac{x}{2}\right)^2}.$$

The function $p_2(x)$ can be used to model the service life of Product A.

(i) Using definite integrals, determine the probability that a randomly selected Product A will have a service life of 1.5 years or less.

(1 mark)

(ii) An approximate measure of the anticipated service life for Product A is m, the median service life of the item, which can be calculated using the equation:

$$\int_0^m p_2(x)\,\mathrm{d}x = 0.5.$$

Using your answer to part (a)(i), explain why the median service life of Product A is greater than 1.5 years.

(1 mark)

(b) (i) If $y = -e^{-\left(\frac{x}{2}\right)^k}$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{2}\left(\frac{x}{2}\right)^{k-1} e^{-\left(\frac{x}{2}\right)^k}$.

(1 mark)

When calculating the median service life of an item, m, using the probability density function $p_k(x)$, the following equation can be used:

$$\int_0^m p_k(x)\,\mathrm{d}x = 0.5.$$

(ii) Determine the *exact* value of k that would result in $p_k(x)$ having a median service life of 1.5 years.

(5 marks)