Stage 2 Specialist Mathematics

Rates of Change and Differential Equations Test

Topic 6: Subtopics 6.1, 6.2, 6.3, 6.4, 6.5

Total Marks - 38

(Calculator and one A4 page of handwritten notes permitted.)

QUESTION 1 (8 marks)

A curve has the following parametric equations:

$$\begin{cases} x(t) = \sin 2t \\ y(t) = \cos^2 t \end{cases} \text{where } 0 \le t \le 2\pi.$$

(a) Sketch a graph of the curve on the axes in Figure 1.

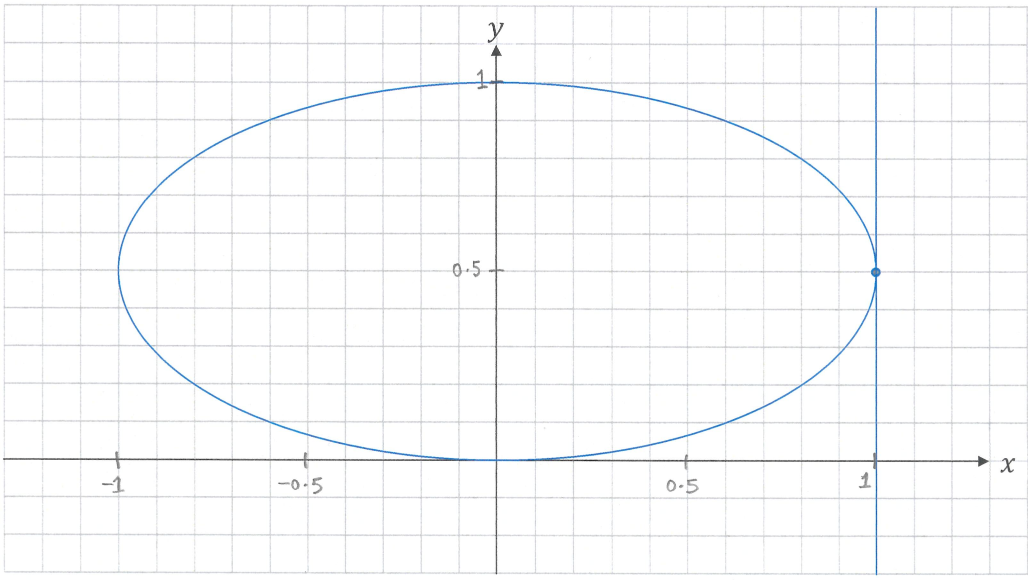

Figure 1

(3 marks)

(b) Show that $\frac{dy}{dx} = -\frac{1}{2}\tan 2t$.

$x'(t) = 2\cos 2t$

$y'(t) = 2\cos t \,.\, -\sin t$

$= -\sin 2t$

$\frac{dy}{dx} = \frac{-\sin 2t}{2\cos 2t}$

$= -\frac{1}{2}\tan 2t$

(3 marks)

(c) (i) What is the slope of the tangent to the curve at $t = \frac{\pi}{4}$?

$t = \frac{\pi}{4} \Rightarrow \frac{dy}{dx} = \text{undefined}$

(1 mark)

(ii) Draw the tangent to the curve at $t = \frac{\pi}{4}$ on your graph in Figure 1.

(1 mark)

QUESTION 2 (8 marks)

Sociologists can study the spread of a new fashion by modelling the rate at which the fashion spreads. For one such model the rate of spread is given by the differential equation

$$\frac{dN}{dt} = \frac{0.5N(10000 - N)}{10000}$$

where N is the number of people who follow the fashion and t is in weeks. Initially there are 100 people following the fashion.

(a) Show that $\frac{10000}{N(10000-N)} = \frac{1}{N} + \frac{1}{10000-N}$.

$$\frac{1}{N} + \frac{1}{10000-N} = \frac{10000 - \cancel{N} + \cancel{N}}{N(10000-N)}$$

$$= \frac{10000}{N(10000-N)}$$

(1 mark)

(b) Solve the differential equation given above with the initial condition $t = 0$, $N = 100$ to show that

$$N = \frac{10000}{1 + 99e^{-0.5t}}$$

$$\frac{dN}{dt} = \frac{0.5N(10000-N)}{10000}$$

$$-\int \frac{10000}{N(10000-N)}\, dN = \int -0.5\, dt$$

$$-\int \frac{1}{N} + \frac{1}{10000-N}\, dN = -0.5t + c$$

$$-\ln|N| + \ln|10000-N| = -0.5t + c$$

$$\ln\left|\frac{10000-N}{N}\right| = -0.5t + c$$

$$\frac{10000}{N} - 1 = Ae^{-0.5t}$$

$$\frac{10000}{N} = 1 + Ae^{-0.5t}$$

$$N = \frac{10000}{1+Ae^{-0.5t}}$$

When $t=0$, $N = 100$

$$\therefore\ 100 = \frac{10000}{1+A}$$

$$1 + A = 100$$

$$A = 99$$

$$\therefore\ N = \frac{10000}{1+99e^{-0.5t}}$$

(5 marks)

(c) What is the estimated time at which the fashion is spreading at the greatest rate?

The fashion is spreading at the greatest rate at the inflection point ie. when $N = \frac{10000}{2}$

$$\therefore \frac{10000}{2} = \frac{10000}{1+99e^{-0.5t}}$$

$$1+99e^{-0.5t} = 2$$

$$99e^{-0.5t} = 1$$

$$e^{-0.5t} = \frac{1}{99}$$

$$-0.5t = -\ln 99$$

$$t = 2\ln 99 \simeq 9.19 \text{ weeks}$$

(2 marks)

QUESTION 3 (7 marks)

A pharmaceutical company markets an antibiotic tablet that has the shape of a cylinder with hemispherical ends, as shown in Figure 2. The surface area of the tablet is 200 square millimetres. The cylindrical section has a length of l millimetres and a radius of r millimetres.

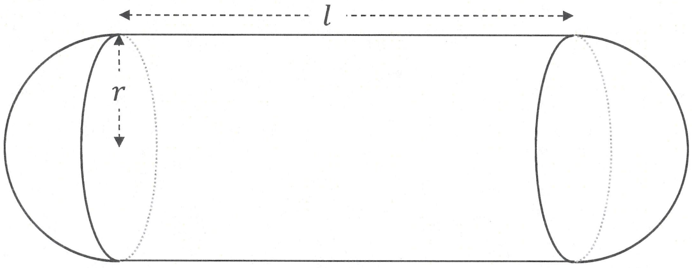

Figure 2

(a) (i) Show that the surface area of the tablet is $A = 2\pi rl + 4\pi r^2$.

$$A = \underbrace{2\pi r \times l}_{\text{cylinder}} + \underbrace{2 \times 2\pi r^2}_{\text{two hemispherical ends}}$$

$$= 2\pi r l + 4\pi r^2$$

(1 mark)

(ii) Hence show that $\frac{dA}{dt} = (2\pi l + 8\pi r)\frac{dr}{dt} + 2\pi r\frac{dl}{dt}$.

Differentiating $A = 2\pi r l + 4\pi r^2$ with respect to t gives

$$\frac{dA}{dt} = 2\pi r\frac{dl}{dt} + 2\pi l\frac{dr}{dt} + 8\pi r\frac{dr}{dt}$$

$$= \left(2\pi l + 8\pi r\right)\frac{dr}{dt} + 2\pi r\frac{dl}{dt}$$

(3 marks)

At a particular instant when the tablet is dissolving:

- the radius is 1 millimetre and is decreasing at the rate of 0.05 millimetres per second;
- the surface area is half its original value and is decreasing at the rate of 6 square millimetres per second.

(b) Find the rate at which the length is changing at this instant.

Substituting the conditions above into the equation for A gives

$$100 = 2\pi l + 4\pi$$

$$2\pi l = 100 - 4\pi$$

$$l = \frac{100 - 4\pi}{2\pi} \simeq 13.9\text{ mm}$$

Substituting the conditions above and this value of l into the equation for $\frac{dA}{dt}$ gives

$$-6 = \left(100 - 4\pi + 8\pi\right).-0.05 + 2\pi.\frac{dl}{dt}$$

$$-6 = -5 - 0.2\pi + 2\pi\frac{dl}{dt}$$

$$2\pi\frac{dl}{dt} = 0.2\pi - 1$$

$$\frac{dl}{dt} = \frac{0.2\pi - 1}{2\pi} \simeq -0.0592\text{ mm/s}$$

(3 marks)

QUESTION 4 (7 marks)

Figure 3 shows the slope field for the differential equation $\frac{dy}{dx} = 3y - 6xy$ and the point $P(0, 1)$.

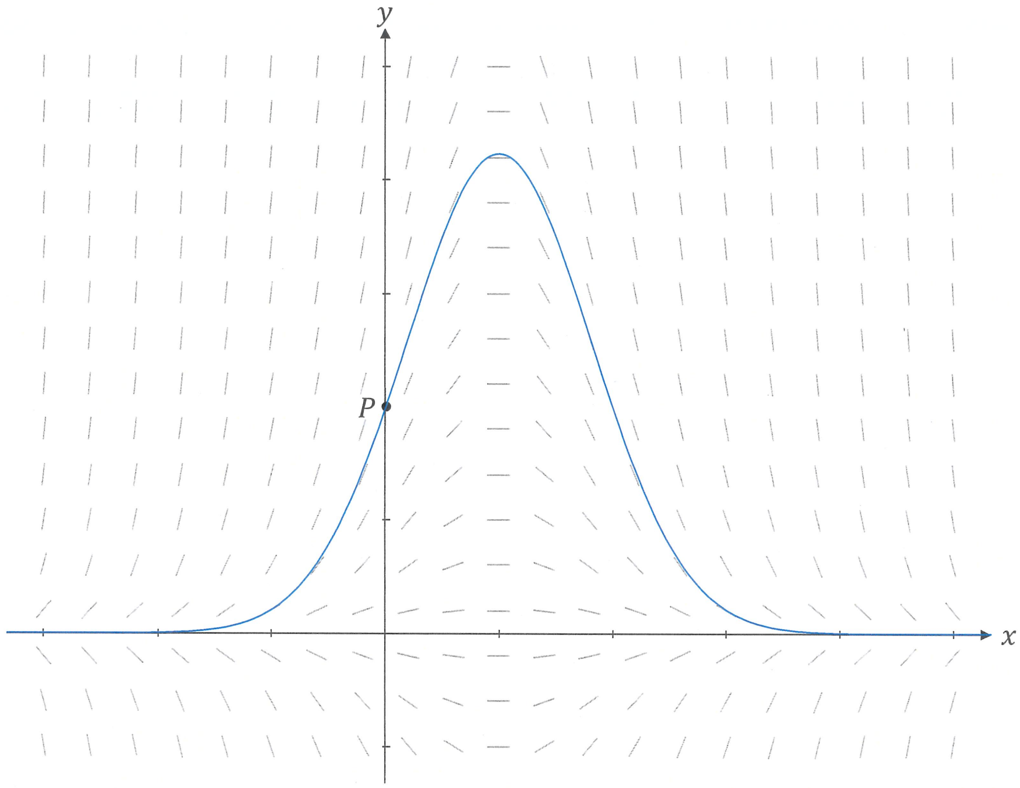

Figure 3

(a) On Figure 3 draw the solution curve that passes through P.

(3 marks)

(b) By solving the differential equation $\frac{dy}{dx} = 3y - 6xy$, with the initial condition $x = 0$ and $y = 1$, show that $y = e^{3x(1-x)}$.

$\frac{dy}{dx} = 3y(1 - 2x)$

$\int \frac{1}{y}\, dy = \int 3 - 6x \, dx$

$\ln|y| = 3x - 3x^2 + c$

$y = Ae^{3x(1-x)}$

When $x = 0$, $y = 1$

$\therefore 1 = A$

$y = e^{3x(1-x)}$

(4 marks)

QUESTION 5 (8 marks)

A Bézier curve has control points $S(8,-7)$, $C_1(1,6)$, $C_2(-4,12)$, and $E(-7,3)$, and for this curve $y(t) = -8t^3 - 21t^2 + 39t - 7$.

(a) Find $x(t)$ for the Bézier curve.

$a_x = (-7) - 3(-4) + 3(1) - (8) = 0$

$b_x = 3(-4) - 6(1) + 3(8) = 6$

$c_x = 3(1) - 3(8) = -21$

$d_x = (8) = 8$

$x(t) = 6t^2 - 21t + 8$

(2 marks)

(b) Find the coordinates of the highest point on the Bézier curve.

To find the highest point, solve $y'(t) = 0$

$-24t^2 - 42t + 39 = 0$

Applying the quadratic formula gives $\cancel{t = -2.42}$ or $t = 0.671$

$(0 \le t \le 1)$

$x(0.671) = -3.39$ and $y(0.671) = 7.30$

ie. the coordinates are $(-3.39, 7.30)$

(4 marks)

(c) Find the x-intercept for the Bézier curve correct to three significant figures.

To find the x-intercept, solve $y(t) = 0$

Using technology, $\cancel{t = -3.92}$ or $t = 0.204$ or $\cancel{t = 1.10}$ $(0 \le t \le 1)$

$x(0.204) = 3.97$ and $y(0.204) = 0$ (as expected)

ie. the coordinates are $(3.97, 0)$

(2 marks)