South Australian
Certificate of Education

1

Specialist Mathematics
2023

Question booklet 1

Questions 1 to 7 (55 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on page 16 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

Examination information

Materials
- Question booklet 1
- Question booklet 2
- Formula sheet
- SACE registration number label

Instructions
- Show appropriate working and steps of logic in the question booklets
- State all answers correct to three significant figures, unless otherwise instructed
- Use black or blue pen
- You may use a sharp dark pencil for diagrams

Total time: 130 minutes
Total marks: 100

Attach your SACE registration number label here

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Question 1 (5 marks)

Consider the plane $P: x+2y-z=4$ and the line l defined by the following parametric equations, where t is a real parameter.

$$\begin{cases} x=1+t \\ y=2-2t \\ z=3-t \end{cases}$$

(a) Show that the point of intersection of P and l is $A(0,4,4)$.

(3 marks)

(b) Find the equation of the plane which passes through A and is perpendicular to l.

(2 marks)

Question 2 (7 marks)

The toy train shown in Figure 1 starts at point B and moves along a straight track.

The rotating camera at point C is 50 cm from B such that CB is perpendicular to the track.

The rotating camera films the train as it moves away from B, making an angle θ as shown.

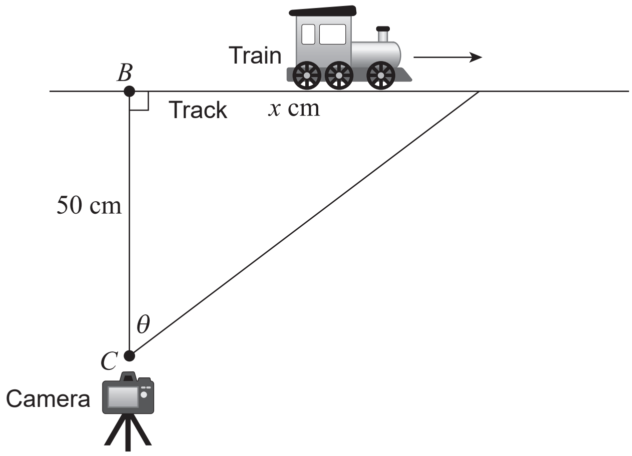

Figure 1

The displacement of the train from B is given by $x = 50\ln(t+1)$, where $t \geq 0$ is measured in seconds and x is measured in centimetres.

(a) Find the exact value of x at $t = 9$ seconds.

(1 mark)

(b) (i) Show that $\theta = \arctan\left(\dfrac{x}{50}\right)$.

(1 mark)

(ii) Hence show that $\dfrac{d\theta}{dt} = \left(\dfrac{50}{2500 + x^2}\right)\dfrac{dx}{dt}$.

(2 marks)

(c) Hence find the rate of change, $\dfrac{d\theta}{dt}$, when $t = 9$ seconds. Give the answer correct to three significant figures.

(3 marks)

Question 3 (7 marks)

(a) Write the following complex numbers in exact polar form.

(i) $z_1 = \sqrt{3} - 3i$

(1 mark)

(ii) $z_2 = 3\sqrt{3} - 3i$

(1 mark)

(b) Complex numbers z_1 and z_2 from part (a) are shown on the Argand diagram in Figure 2. The measure of the acute angle between z_1 and z_2 is θ.

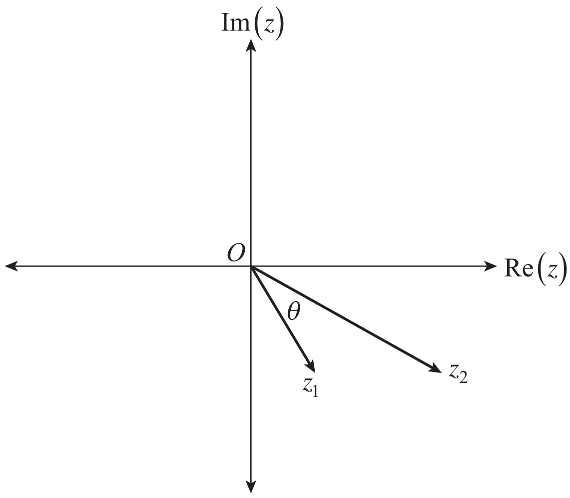

Figure 2

(i) Write $\frac{z_2}{z_1}$ in polar form.

(1 mark)

(ii) State the exact value of θ.

(1 mark)

(c) The complex number z_2 from part (a) is scaled by a factor of $\frac{1}{2}$ and rotated anticlockwise about the origin O through $\frac{3\theta}{2}$ to produce complex number z_3, as shown on the Argand diagram in Figure 3.

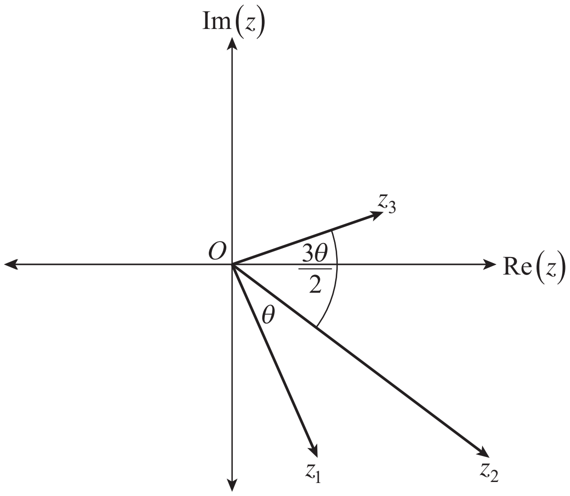

Figure 3

Using the value of θ found in part (b)(ii), find z_3 in exact polar form.

(3 marks)

Question 4 (7 marks)

(a) Prove by mathematical induction that $4^n + 15n - 1$ is divisible by 9 for all positive integers n.

(5 marks)

(b) Using the result of part (a), show that $4^{n+1} + 60n - 1$ is divisible by 3 for all positive integers n.

(2 marks)

Question 5 (9 marks)

Consider the planes described by the following system of equations where a is a real constant.

$$\begin{cases} x+y-3z=4 \\ x-y-az=2 \\ 3x+ay-z=8 \end{cases}$$

(a) (i) Write the system of equations in augmented matrix form.

(1 mark)

(ii) Using clearly defined row operations, show that the system can be reduced to:

$$\begin{bmatrix} 1 & 1 & -3 & : & 4 \\ 0 & 2 & (a-3) & : & 2 \\ 0 & 0 & (a-7)(a+1) & : & 2(a+1) \end{bmatrix}$$

(3 marks)

(b) For this system of equations, what value of a corresponds to the geometric configuration of the three planes shown in Figure 4?

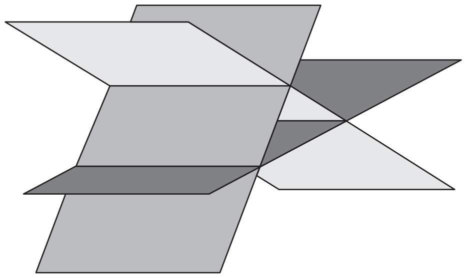

Figure 4

(1 mark)

(c) Interpret the geometric solution for the case $a = 3$.

(1 mark)

(d) (i) State the value of a for which the intersection of the three planes is a line.

(1 mark)

(ii) Find the equation of this line for the value of a found in part (d)(i).

(2 marks)

Question 6 (10 marks)

Source: © Jen567 | Pixabay.com

The population B of butterflies in a butterfly house has a rate of growth, $\frac{dB}{dt}$, that is modelled by the differential equation

$$\frac{dB}{dt} = 0.1B\left(1 - \frac{B}{K}\right),$$

where K is a positive constant and t is the time in months, $t \geq 0$, and B equals the number of butterflies in the population.

(a) (i) Show that $\frac{K}{B(K-B)} = \frac{1}{B} + \frac{1}{K-B}$.

(1 mark)

(ii) Using integration techniques, show that the butterfly population can be modelled by $B = \frac{K}{1 + Ae^{-0.1t}}$ for some constant A.

(4 marks)

(b) For a particular population of butterflies in a butterfly house with an initial population of 120 butterflies (as marked), sketch the solution curve on the slope field in Figure 5 below.

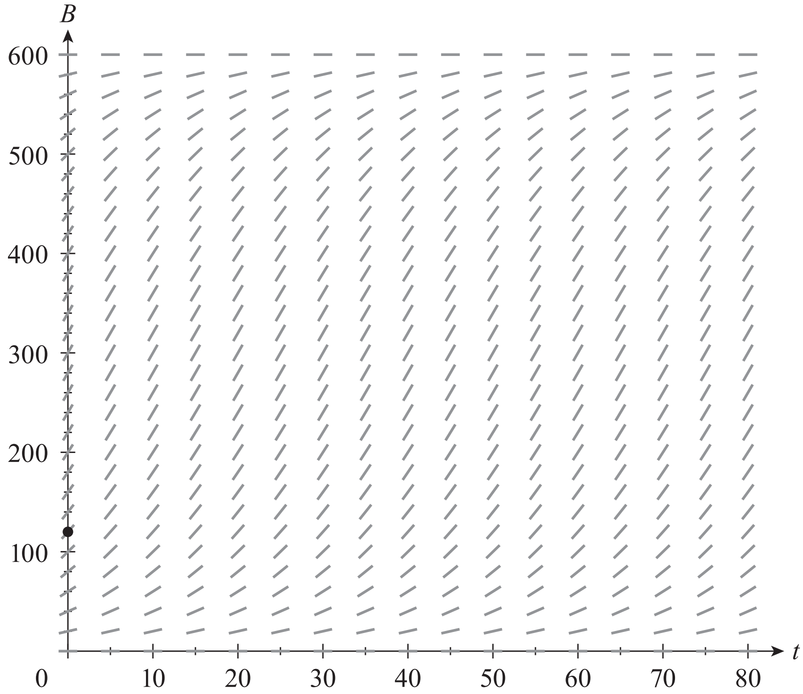

Figure 5

(2 marks)

(c) The butterfly house modelled in part (b) has undergone extensions and can now accommodate up to 960 butterflies. A growing butterfly population with an initial population of 120 butterflies is introduced to the butterfly house. Use the information stated in part (a)(ii) to answer the following.

(i) State the value of K.

(1 mark)

(ii) Find the value of A.

(1 mark)

(iii) How long will it take for this population to exceed 600 butterflies?

(1 mark)

Question 7 (10 marks)

(a) Given that $\int xe^x \mathrm{d}x = xe^x - e^x + c$, use integration by parts to show that $\int x^2 e^x \mathrm{d}x = e^x\left(x^2 - 2x + 2\right) + k$, where c and k are constants.

(2 marks)

(b) Consider the functions $f(x) = \sqrt{x}$ and $g(x) = \frac{e^x}{4}\left(4 - x^2\right)$.

(i) Show that $f(g(x)) = \frac{1}{2}\sqrt{e^x\left(4 - x^2\right)}$.

(1 mark)

(ii) On Figure 6 below draw the graph of $y = f(g(x)) = \frac{1}{2}\sqrt{e^x\left(4 - x^2\right)}$ for $-2 \le x \le 2$.

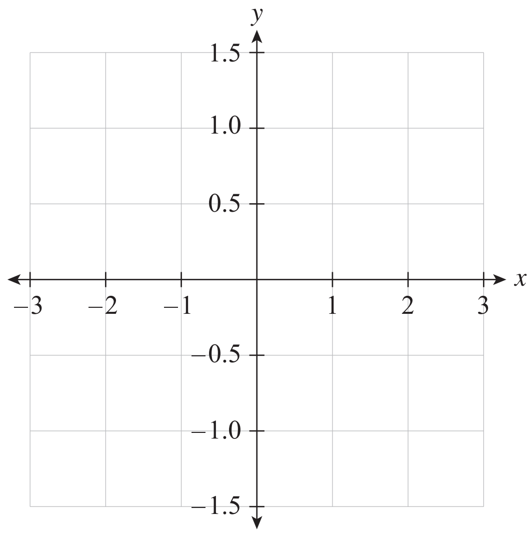

Figure 6 (3 marks)

(c) The curve $y = f\left(g\left(x\right)\right)$ sketched in Figure 6 is rotated 2π radians about the x axis for $-2 \le x \le 2$ to form a solid.

(i) Show that the volume of the solid formed by the rotation is given by

$$V = \frac{\pi}{4}\int_{-2}^{2} e^{x}\left(4 - x^{2}\right)\mathrm{d}x.$$

(1 mark)

(ii) Using part (a), show that the volume of the solid is $\frac{\pi}{2}\left(e^{2} + 3e^{-2}\right)$.

(3 marks)

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 5(a)(ii) continued).

South Australian Certificate of Education

Specialist Mathematics
2023

Question booklet 2

Questions 8 to 10 (45 marks)
- Answer ***all*** questions
- Write your answers in this question booklet
- You may write on page 11 if you need more space
- Allow approximately 65 minutes
- Approved calculators may be used — complete the box below

2

Copy the information from your SACE label here

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Question 8 (15 marks)

(a) Triangle ABT, shown in Figure 7, has $\angle ATB = 90°$ and $\angle ABT = \theta$.

The vector $\boldsymbol{l}$ is in the direction of $\overrightarrow{BT}$, passing through B.

Using $|\overrightarrow{BA} \times \boldsymbol{l}| = |\overrightarrow{BA}||\boldsymbol{l}|\sin\theta$, show that $|\overrightarrow{AT}| = \dfrac{|\overrightarrow{BA} \times \boldsymbol{l}|}{|\boldsymbol{l}|}$.

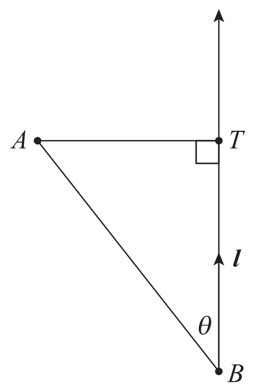


Figure 7

(2 marks)

Point $A(4, 0, 2)$ on plane P_1: $2x + 2y - z = 6$ and point $B(-1, -6, -2)$ on plane P_2: $2x + 2y - z = -12$ are shown in Figure 8.

Point T is on P_1, such that $\overrightarrow{BT}$ is normal to P_1 and to P_2.

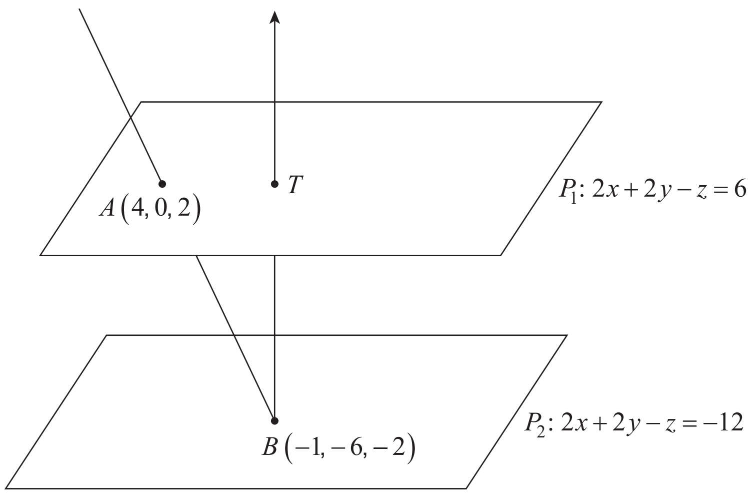


Figure 8

(b) Find $\overrightarrow{BA}$.

(1 mark)

(c) It may be assumed that a normal vector to P_1 and P_2 is $\boldsymbol{l} = [2, 2, -1]$.

(i) Find $|\overrightarrow{BA} \times \boldsymbol{l}|$.

(2 marks)

(ii) Find $|\overrightarrow{AT}|$, using the result of part (a).

(2 marks)

(iii) Find the equation of the line normal to P_2 through $B(-1, -6, -2)$.

(2 marks)

(iv) Show that the line found in part (c)(iii) intersects P_1 at $T(3, -2, -4)$.

(2 marks)

The plane P_3: $2x+2y-z=k$, where k is a constant, is shown in Figure 9.

The line passing through A and B intersects P_3 at point C, as shown.

The line normal to P_3 through C meets P_1 at point V such that $\overrightarrow{AV}=5\,\overrightarrow{AT}$.

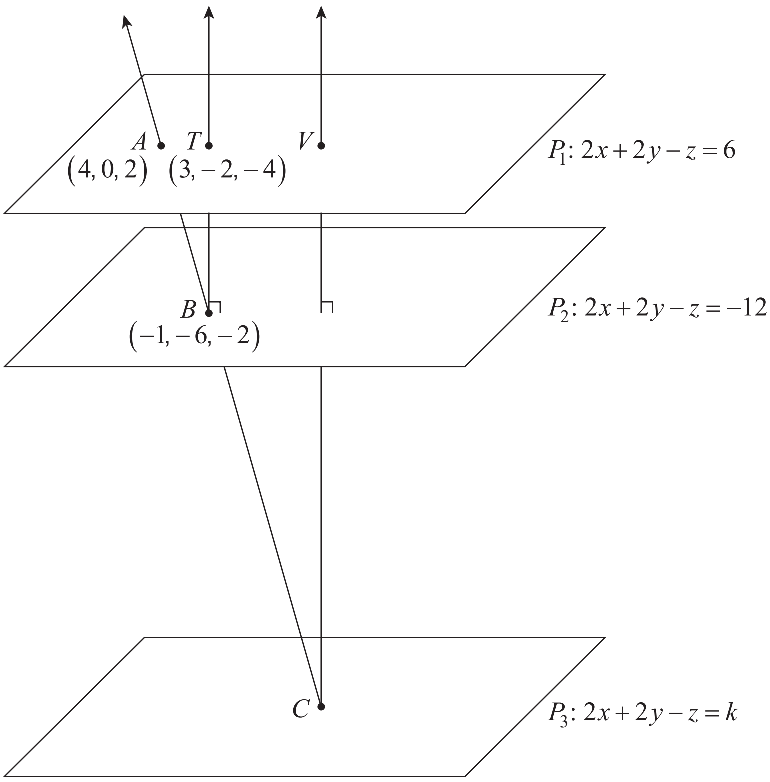


Figure 9

(d) Find the value of k.

(4 marks)

Question 9 (15 marks)

On the Argand diagram in Figure 10 is the set of complex numbers z such that $z = \text{cis}\theta$ for $-\pi < \theta \le \pi$.

The Argand diagram in Figure 11 shows the complex numbers z_1, z_2, z_3, z_4, z_5 which are the zeros of the polynomial $z^5 - 1$.

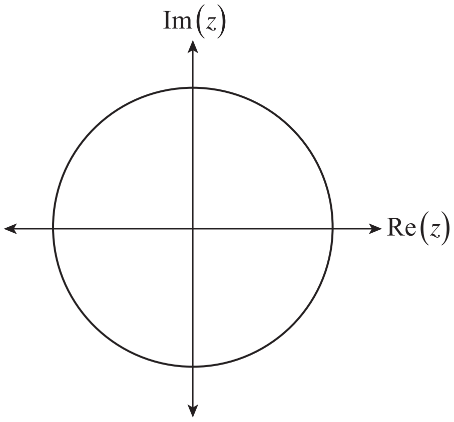


Figure 10

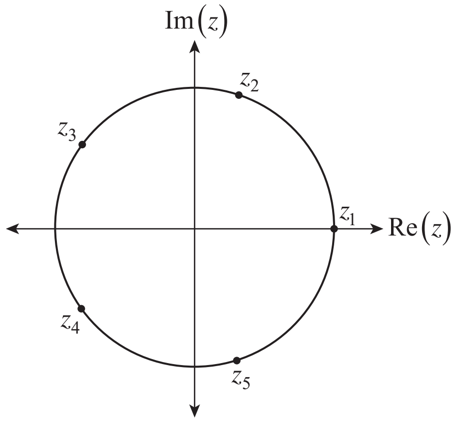


Figure 11

(a) Using De Moivre's theorem or otherwise, write the zeros of $z^5 - 1$ in exact polar form.

(3 marks)

(b) Consider the function $f(z) = z^2 + z + 1 + \dfrac{1}{z} + \dfrac{1}{z^2}$, where $z = \text{cis}\theta$.

(i) Show that $|f(z)| \le 5$.

(2 marks)

(ii) Find a value of z for which $|f(z)| = 5$.

(1 mark)

(c) (i) Show that $z^5 - 1 = z^2(z-1) \times f(z)$.

(2 marks)

(ii) Hence list the four zeros of $f(z)$ in polar form.

(1 mark)

(d) Using $z = \text{cis}\theta$:

(i) show that $f(z)$ is real.

(3 marks)

(ii) show that $f(z) = 4\cos^2\theta + 2\cos\theta - 1$.

(1 mark)

(iii) find the minimum value of $f(z)$.

(1 mark)

(iv) on Figure 12, mark the value(s) of z for which $f(z)$ is the minimum.

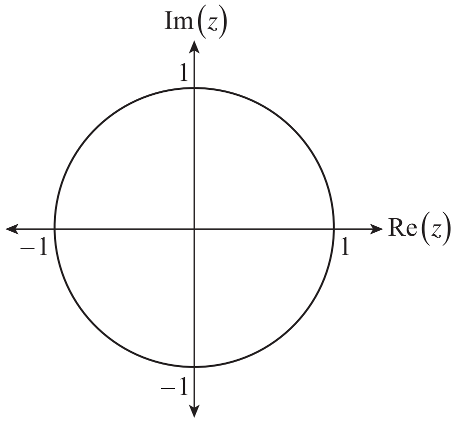


Figure 12

(1 mark)

Question 10 (15 marks)

(a) Show that $\int \cos^2 x \, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$, where c is a constant.

(2 marks)

(b) On Figure 13, draw the curve defined by the following parametric equations, where t is time in seconds.

$$\begin{cases} x = 4\sqrt{2}\sin t \\ y = \frac{\sqrt{2}}{2}\sin 2t \end{cases} \quad \text{for } 0 \le t \le 2\pi$$

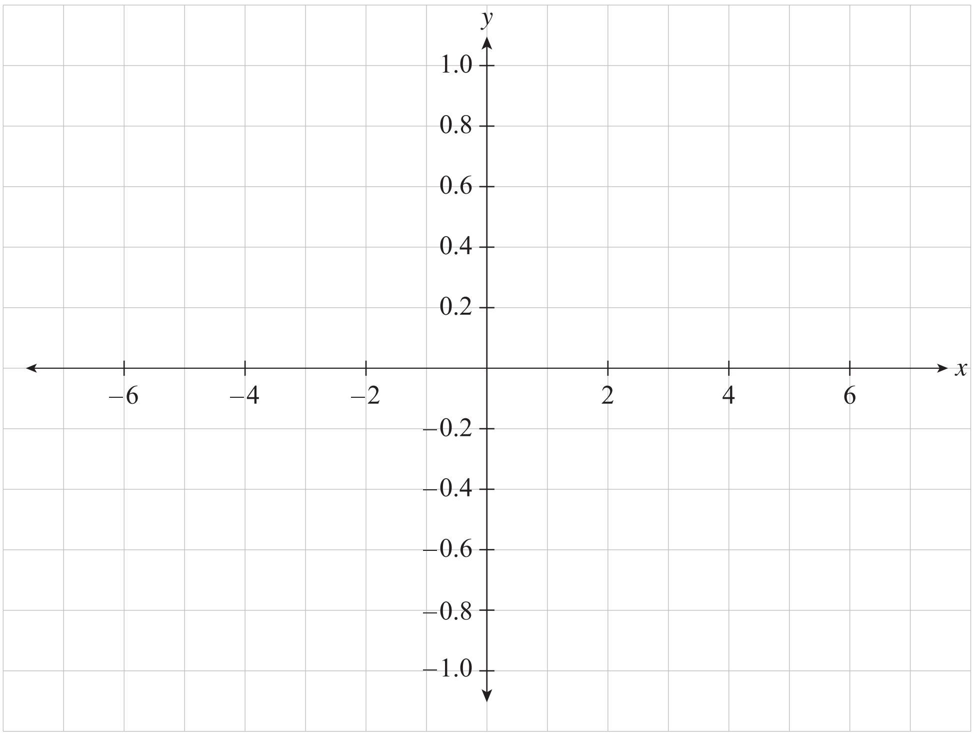


Figure 13

(3 marks)

Consider a dragonfly flying over a pond. The flight path of this dragonfly is modelled in 3D by the parametric equations

$$\begin{cases} x = 4\sqrt{2}\sin t \\ y = \dfrac{\sqrt{2}}{2}\sin 2t \\ z = 4t \end{cases}$$

for $0 \le t \le 2\pi$, where t is time in seconds.

The flight path of this dragonfly is shown in Figure 14 for $0 \le t \le 2\pi$.

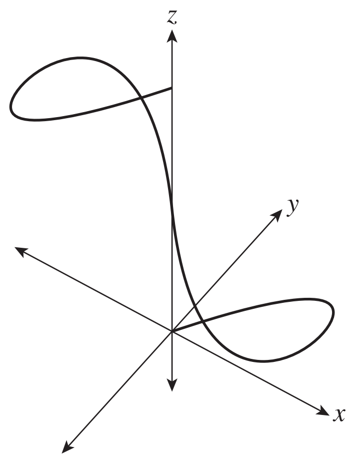


Figure 14

Source: © michel78250 | Pixabay.com

(c) (i) Find the exact velocity vector $\boldsymbol{v}$ for the flight of this dragonfly.

(3 marks)

(ii) Show that the speed of the dragonfly at any time is given by $2\sqrt{2}\cos^2 t + 3\sqrt{2}$.

(4 marks)

Question 10 continues on page 10

(iii) Hence show that the exact distance travelled by the dragonfly over the interval $\frac{\pi}{4} \le t \le \frac{3\pi}{4}$ is $\sqrt{2}\left(2\pi - 1\right)$.

(3 marks)

You may write on this page if you need more space to finish your answers to any of the questions in this question booklet. Make sure to label each answer carefully (e.g. 10(c)(ii) continued).

SPECIALIST MATHEMATICS FORMULA SHEET

Circular functions

$\sin^2 A + \cos^2 A = 1$

$\tan^2 A + 1 = \sec^2 A$

$1 + \cot^2 A = \operatorname{cosec}^2 A$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}$$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

$\sin A \pm \sin B = 2 \sin \frac{1}{2}(A \pm B) \cos \frac{1}{2}(A \mp B)$

$\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$

$\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$

Matrices and determinants

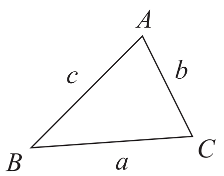
If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = |A| = ad - bc$ and

$A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Measurement

Area of sector, $A = \frac{1}{2} r^2 \theta$, where θ is in radians.
Arc length, $l = r\theta$, where θ is in radians.

In any triangle ABC:



Area of triangle $= \frac{1}{2} ab \sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$a^2 = b^2 + c^2 - 2bc \cos A$

Quadratic equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Distance from a point to a plane

The distance from (x_1, y_1, z_1) to $Ax + By + Cz + D = 0$ is given by

$\dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

Derivatives

$f(x) = y$	$f'(x) = \dfrac{dy}{dx}$
$\arcsin x$	$\dfrac{1}{\sqrt{1 - x^2}}$
$\arccos x$	$\dfrac{-1}{\sqrt{1 - x^2}}$
$\arctan x$	$\dfrac{1}{1 + x^2}$

Properties of derivatives

$\dfrac{d}{dx}\big(f(x)\,g(x)\big) = f'(x)g(x) + f(x)g'(x)$

$\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$

$\dfrac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$

Arc length along a parametric curve

$l = \displaystyle\int_a^b \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}\, dt$, where $a \le t \le b$.

Integration by parts

$\displaystyle\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$

Volumes of revolution

About x axis, $V = \displaystyle\int_a^b \pi y^2\,dx$, where y is a function of x.

About y axis, $V = \displaystyle\int_c^d \pi x^2\,dy$, where y is a one-to-one function of x.