Question 9 (15 marks)

On the Argand diagram in Figure 10 is the set of complex numbers z such that $z = \text{cis}\theta$ for $-\pi < \theta \le \pi$.

The Argand diagram in Figure 11 shows the complex numbers z_1, z_2, z_3, z_4, z_5 which are the zeros of the polynomial $z^5 - 1$.

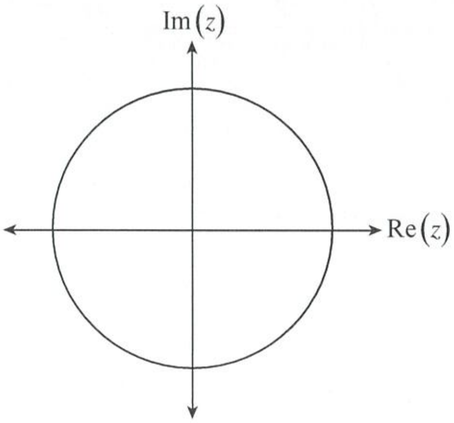

Figure 10

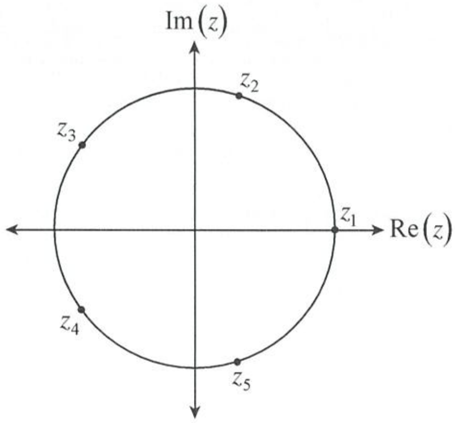

Figure 11

(a) Using De Moivre's theorem or otherwise, write the zeros of $z^5 - 1$ in exact polar form.

$z^5 = \text{cis}(k.2\pi)$ $k = \{0, 1, 2, 3, 4\}$

$z = \text{cis}\left(k.\frac{2\pi}{5}\right)$

$z_1 = \text{cis}(0) = 1$, $z_2 = \text{cis}\left(\frac{2\pi}{5}\right)$, $z_3 = \text{cis}\left(\frac{4\pi}{5}\right)$, $z_4 = \text{cis}\left(\frac{-4\pi}{5}\right)$, $z_5 = \text{cis}\left(\frac{-2\pi}{5}\right)$

(3 marks)

(b) Consider the function $f(z) = z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2}$, where $z = \text{cis}\theta$.

(i) Show that $|f(z)| \le 5$.

$\left|z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2}\right| \le |z^2| + |z| + |1| + \left|\frac{1}{z}\right| + \left|\frac{1}{z^2}\right|$

$= 5$

(2 marks)

(ii) Find a value of z for which $|f(z)| = 5$.

$z = 1$

(1 mark)

(c) (i) Show that $z^5 - 1 = z^2(z-1) \times f(z)$.

$$\begin{aligned}
RHS &= z^2(z-1).f(z) \\
&= (z^3 - z^2)\left(z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2}\right) \\
&= z^5 + \cancel{z^4} + \cancel{z^3} + \cancel{z^2} + \cancel{z} - \cancel{z^4} - \cancel{z^3} - \cancel{z^2} - \cancel{z} - 1 \\
&= z^5 - 1 \\
&= LHS
\end{aligned}$$

(2 marks)

(ii) Hence list the four zeros of $f(z)$ in polar form.

$z = \text{cis}\left(\frac{2\pi}{5}\right), \ \text{cis}\left(\frac{4\pi}{5}\right), \ \text{cis}\left(-\frac{4\pi}{5}\right), \ \text{cis}\left(-\frac{2\pi}{5}\right)$

(1 mark)

(d) Using $z = \text{cis}\theta$:

(i) show that $f(z)$ is real.

$$\begin{aligned}
f(z) &= \text{cis}2\theta + \text{cis}\theta + 1 + \text{cis}(-\theta) + \text{cis}(-2\theta) \\
&= \cos 2\theta + \cancel{i\sin 2\theta} + \cos\theta + \cancel{i\sin\theta} + 1 + \cos\theta - \cancel{i\sin\theta} + \cos 2\theta - \cancel{i\sin 2\theta}
\end{aligned}$$

$\therefore f(z)$ is real

(3 marks)

(ii) show that $f(z) = 4\cos^2\theta + 2\cos\theta - 1$.

$$\begin{aligned}
f(z) &= 2\cos 2\theta + 2\cos\theta + 1 \\
&= 2(2\cos^2\theta - 1) + 2\cos\theta + 1 \\
&= 4\cos^2\theta + 2\cos\theta - 1
\end{aligned}$$

(1 mark)

(iii) find the minimum value of $f(z)$.

Using technology, min value of $f(z) = -1.25$

(1 mark)

(iv) on Figure 12, mark the value(s) of z for which $f(z)$ is the minimum.

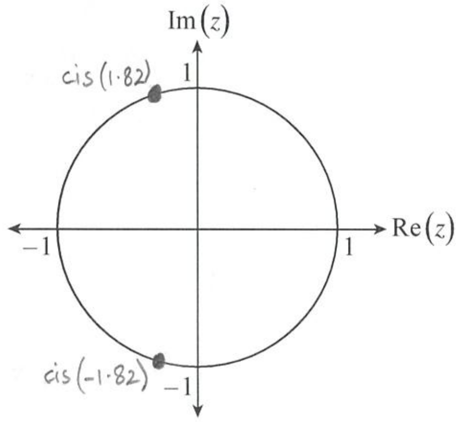

Figure 12

(1 mark)