Stage 2 Mathematical Methods

Differential Calculus Test

Topic 1: Subtopics 1.1, 1.2, 1.5

Total Marks – 59

This Skills and Applications Task is to be completed without a calculator or notes.

Question 1 (10 marks)

Differentiate the following. There is no need to simplify your answers.

(a) $y = 2x^2 - 6\sqrt{x}$

$y = 2x^2 - 6x^{1/2}$

$\frac{dy}{dx} = 4x - 3x^{-1/2}$

(2 marks)

(b) $y = \frac{x^2-4x+5}{x^2}$

$y = 1 - 4x^{-1} + 5x^{-2}$

$\frac{dy}{dx} = 4x^{-2} - 10x^{-3}$

(2 marks)

(c) $y = (3x - 2x^4)^5$

$\frac{dy}{dx} = 5(3x-2x^4)^4.(3-8x^3)$

(2 marks)

(d) $f(x) = 3x^2(5-2x)^3$

$f'(x) = 6x.(5-2x)^3 + 3x^2.3(5-2x)^2.-2$

(2 marks)

(e) $y = \frac{4\sqrt{x}}{(6+2x)^2} = \frac{4x^{1/2}}{(6+2x)^2}$

$\frac{dy}{dx} = \frac{2x^{-1/2}.(6+2x)^2 - 4x^{1/2}.2(6+2x).2}{(6+2x)^4}$

(2 marks)

Question 2 (5 marks)

Find, from first principles, $f'(6)$ if $f(x) = \frac{x}{x-4}$

$$f'(6) = \lim_{h \to 0} \frac{f(6+h) - f(6)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{6+h}{2+h} - 3}{h}$$

$$= \lim_{h \to 0} \frac{\frac{6+h}{2+h} - \frac{3(2+h)}{2+h}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{6} + h - \cancel{6} - 3h}{h(2+h)}$$

$$= \lim_{h \to 0} \frac{-2\cancel{h}}{\cancel{h}(2+h)}$$

$$= \frac{-2}{2}$$

$$= -1$$

(5 marks)

Question 3 (4 marks)

Find the equation, in the form $y = mx + c$, of the **TANGENT** to $y = 8\sqrt{x} - \frac{1}{x^2}$ at the point where $x = 1$.

$y(1) = 8 - 1 = 7$

$y(x) = 8x^{1/2} - x^{-2}$

$y'(x) = 4x^{-1/2} + 2x^{-3}$

$= \frac{4}{\sqrt{x}} + \frac{2}{x^3}$

$y'(1) = 4 + 2 = 6$

The equation of the tangent is $y = 6(x-1) + 7$

$y = 6x - 6 + 7$

$y = 6x + 1$

(4 marks)

Question 4 (14 marks)

The function $f(x) = -3x^4 + 2ax^3 - 24x^2 + 6$ has a stationary point at $x = 2$.

(a) Explain why $f'(2) = 0$.

Whenever $f(x)$ has a stationary point, $f'(x) = 0$

$f(x)$ has a stationary point when $x = 2$ $\therefore f'(2) = 0$

(1 mark)

(b) Hence show that $a = 8$.

$f'(x) = -12x^3 + 6ax^2 - 48x$

$= -6x(2x^2 - ax + 8)$

$f'(2) = 0 \Rightarrow \cancel{-12}(8 - 2a + 8) = 0$

$16 - 2a = 0$

$-2a = -16$

$a = 8$

(2 marks)

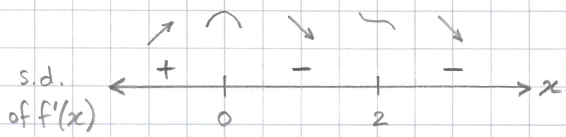
(c) Find and classify the stationary points of $f(x)$. (Make sure you include a sign diagram.)

$f(x) = -3x^4 + 16x^3 - 24x^2 + 6$

$f'(x) = -12x^3 + 48x^2 - 48x$

$= -12x(x^2 - 4x + 4)$

$= -12x(x-2)^2$



$f(0) = 6$

$f(2) = -48 + 128 - 96 + 6 = -10$

- $f(x)$ has a local maximum at $(0, 6)$
- $f(x)$ has a stationary inflection point at $(2, -10)$

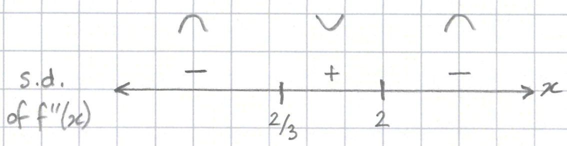
(4 marks)

(d) Find the x-coordinates of the points of inflection of $f(x)$ and use a sign diagram to determine what shape change is occurring at these points.

$f''(x) = -36x^2 + 96x - 48$

$= -12(3x^2 - 8x + 4)$

$= -12(3x-2)(x-2)$



- when $x = 2/3$, the shape of $f(x)$ is changing from concave downwards to concave upwards
- when $x = 2$, the shape of $f(x)$ is changing from concave upwards to concave downwards

(3 marks)

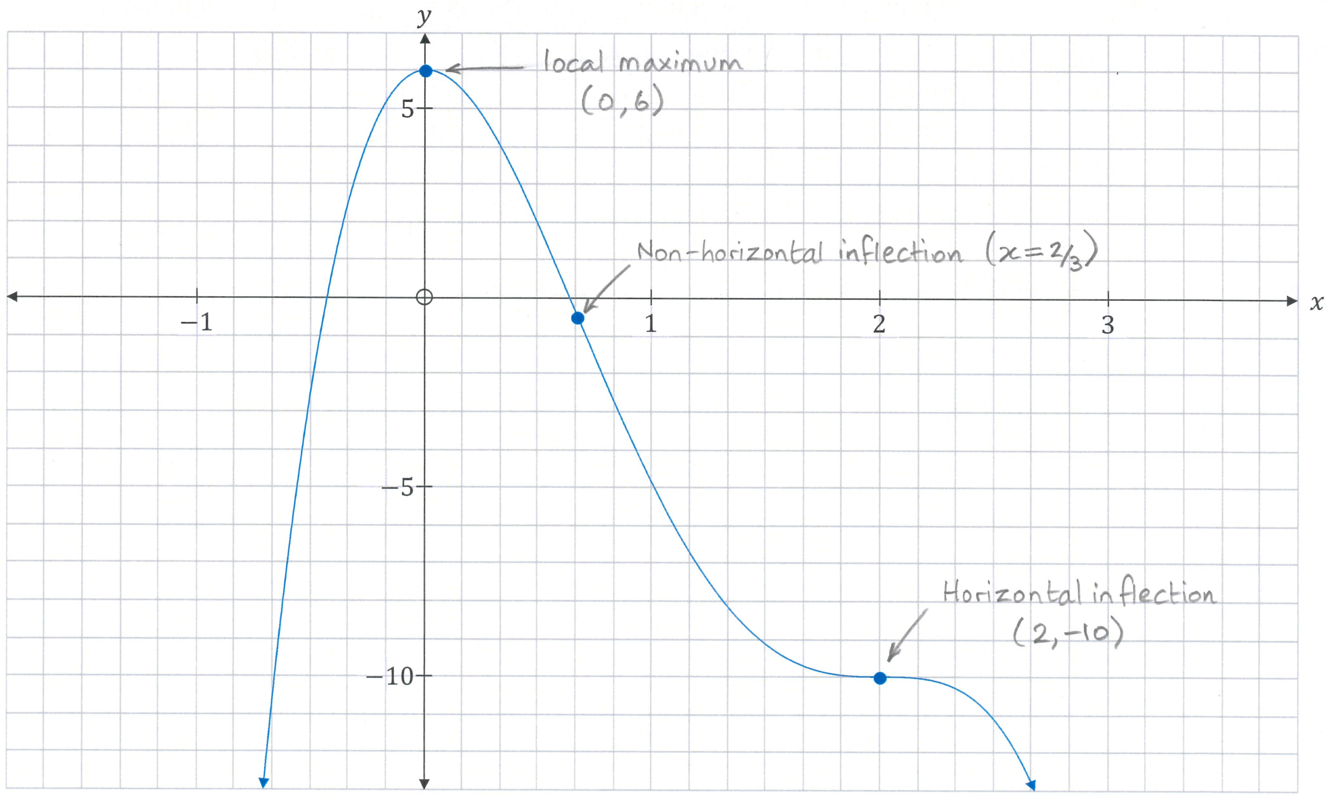
(e) Classify the points you found in part (d) as horizontal or non-horizontal inflection points.

From the sign diagram of $f'(x)$ in part (c) we see that $f'(2/3) \neq 0$ and $f'(2) = 0$

$\therefore$ Non-horizontal inflection point when $x = \frac{2}{3}$. Horizontal inflection point when $x = 2$

(2 marks)

(f) Sketch the graph of $y = f(x)$ labelling all the information from parts (c), (d) and (e).



(2 marks)

Question 5 (17 marks)

A particle moves in a straight line such that its position t seconds after it has passed through the origin, O, is given by $s(t) = 2t^3 - 9t^2 + 12t - 5$ metres, $t \geq 0$.

(a) Find expressions for the velocity and acceleration of the particle after t seconds.

$v(t) = 6t^2 - 18t + 12$ m/s

$a(t) = 12t - 18$ m/s^2

(2 marks)

(b) Find when the particle is at rest and its position at these times.

$v(t) = 0 \Rightarrow 6t^2 - 18t + 12 = 0$

$6(t^2 - 3t + 2) = 0$

$6(t-1)(t-2) = 0$

$t = 1$ or $t = 2$ seconds

$s(1) = 2 - 9 + 12 - 5 = 0$ ie. at the origin

$s(2) = 16 - 36 + 24 - 5 = -1$ ie. 1m to the left of the origin

(4 marks)

(c) Hence or otherwise, determine the time when the particle **PASSES THROUGH** the origin.

$s(t)$ and $v(t)$ are both zero when $t = 1$

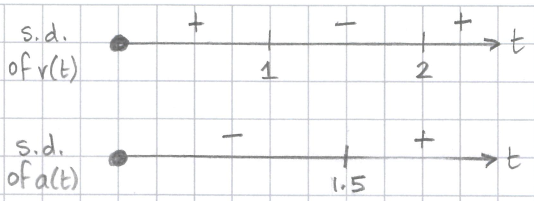
$\therefore (t-1)^2$ is a factor of $s(t)$

$s(t) = (t-1)^2(2t-5)$

$\therefore$ The particle passes through the origin when $t = 2{\cdot}5$ seconds

(1 mark)

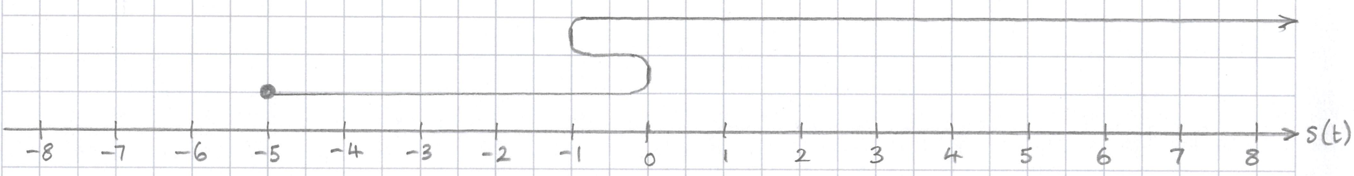
(d) Draw sign diagrams for $v(t)$ and $a(t)$ and determine the time(s) when the particle's speed is **INCREASING**.



The particle's speed is increasing when $1 \leq t \leq 1.5$ and $t \geq 2$ seconds

(4 marks)

(e) Draw a diagram to illustrate the motion of the particle.



(2 marks)

(f) Calculate the total distance travelled by the particle in the first 3 seconds of its motion.

$s(0) = -5$

5m

$s(1) = 0$

1m

$s(2) = -1$

5m

$s(3) = 54 - 81 + 36 - 5 = 4$

Total distance travelled $= 11\text{m}$

(4 marks)

Question 6 (9 marks)

Consider the function $h(t) = 3\left(2 - \frac{8}{t+6}\right)$ where $h(t)$ is the height of a tree t years after it was planted.

(a) Find the height of the tree at the time that it was planted.

$h(0) = 3\left(2 - \frac{4}{3}\right)$

$= 2m$

(1 mark)

(b) Determine how much the tree had grown in the first 3 years of its life.

$h(3) = 3\left(2 - \frac{8}{9}\right) = 3\frac{1}{3}\ m$

$h(3) - h(0) = 1\frac{1}{3}$ m of growth

(2 marks)

(c) Find the rate at which the tree is growing after 5 years.

$h(t) = 6 - 24(t+6)^{-1}$

$h'(t) = 24(t+6)^{-2} \cdot 1$

$h'(5) = \frac{24}{121}$ m/year

(2 marks)

(d) Is $h'(t) > 0$ for all values of t? What is the significance of this result?

$h'(t) = \frac{24}{(t+6)^2} > 0$ for all $t \geq 0$

$\therefore$ The tree is always growing

(2 marks)

(e) Is $h''(t) > 0$ for all values of t? What is the significance of this result?

$h''(t) = \frac{-48}{(t+6)^3} < 0$ for all $t \geq 0$

$\therefore$ The rate at which the tree is growing is always decreasing

(2 marks)